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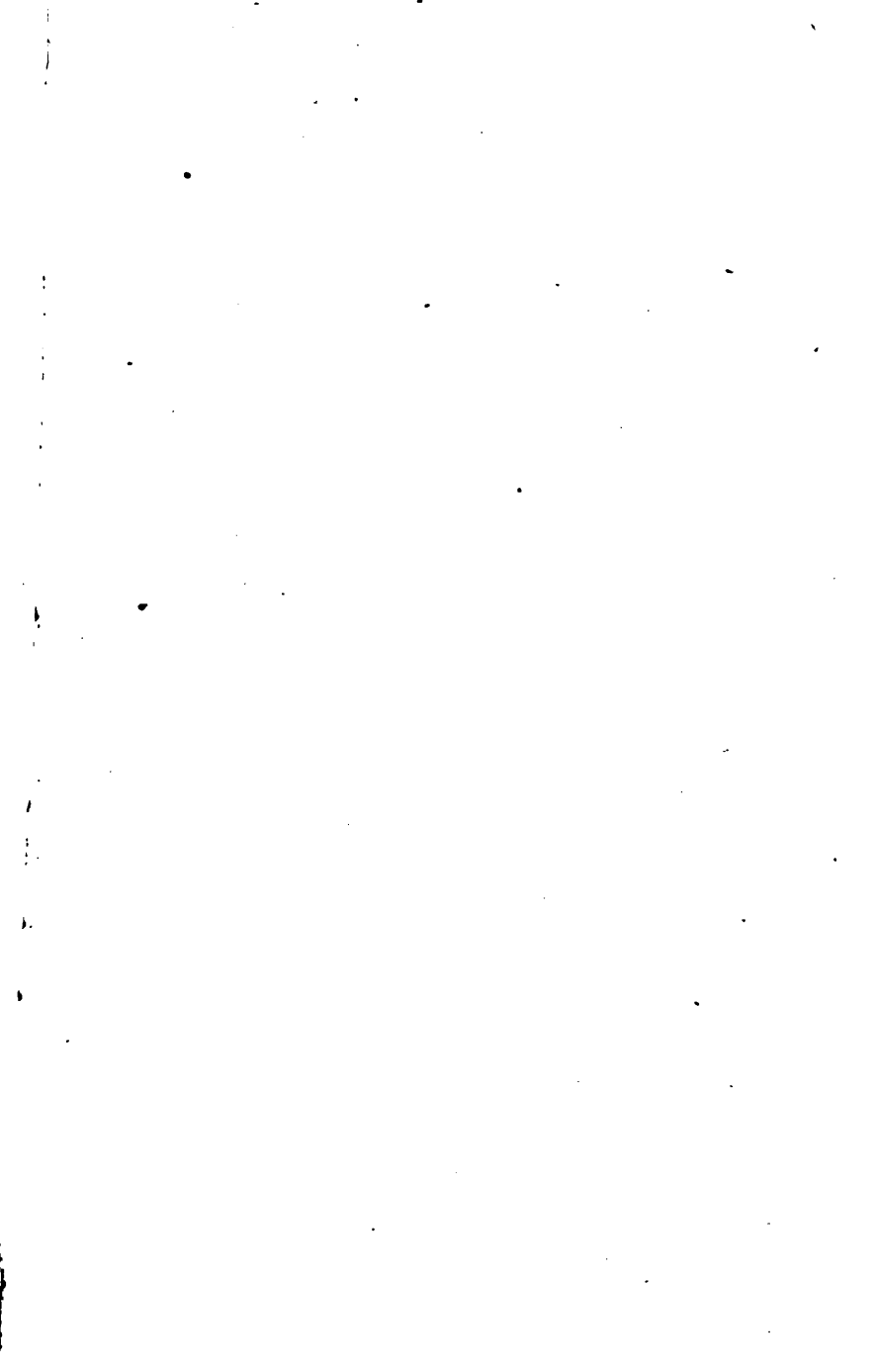


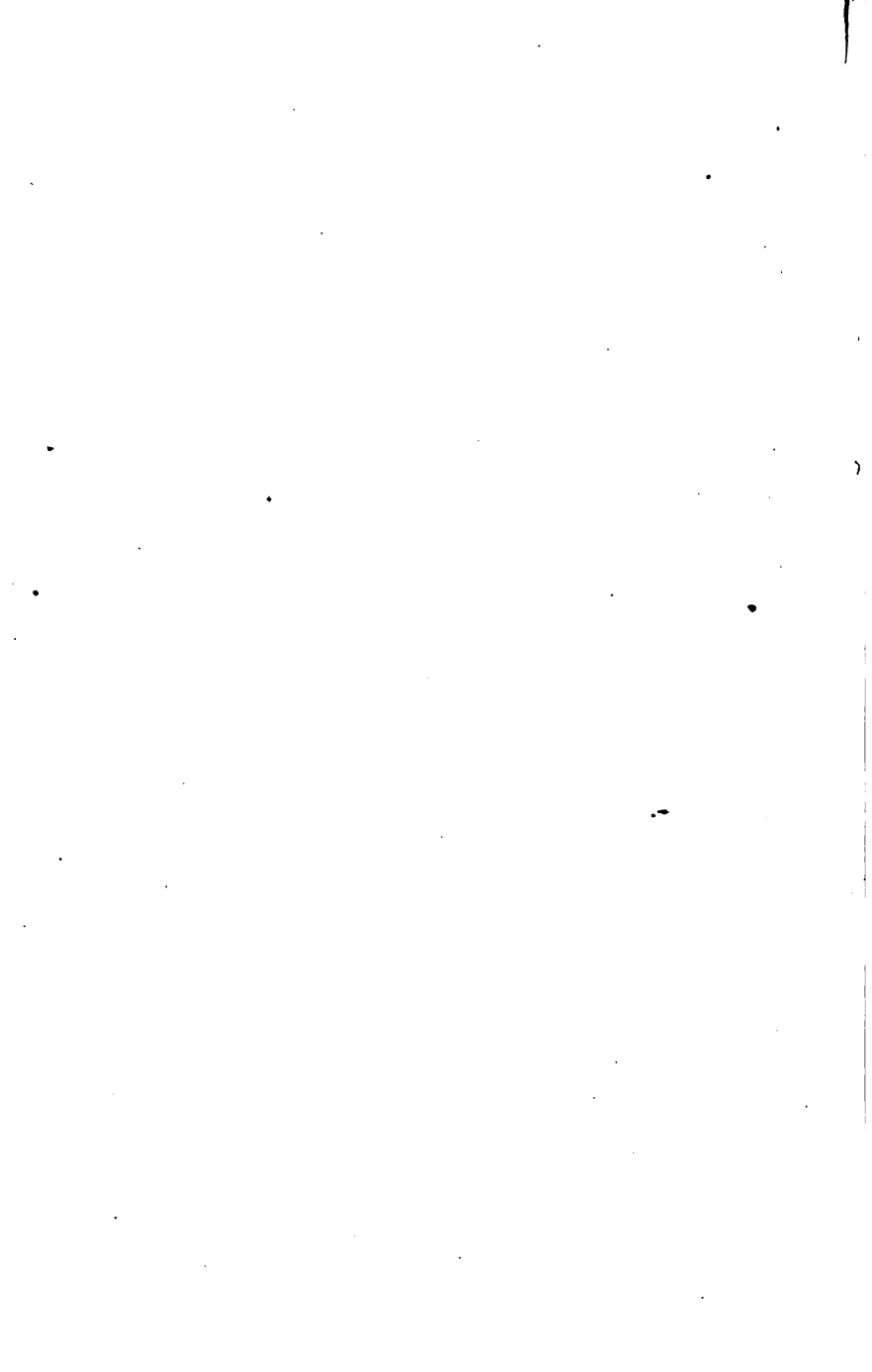
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# PRACTICAL ALGEBRA:

ADAPTED TO

THE IMPROVED METHODS OF INSTRUCTION

IN

SCHOOLS, ACADEMIES, AND COLLEGES

WITH AN APPENDIX.

BY

JAMES B. THOMSON, LL. D.,

AUTHOR OF A SERIES OF MATHEMATICS.

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1888.

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## P R E F A C E.

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**I**T has long been a favorite plan of the author to make a Practical Algebra—a Book combining the important principles of the Science, with their application to methods of business.

Several years have elapsed since he began to gather and arrange materials for this object. Many of the more important parts have been written and re-written and again revised, till they have found embodiment in the book now offered to the public.

In the execution of this plan, clearness and brevity in the definitions and rules have been the constant aim.

A series of practical problems, applying the principles already explained, has been introduced into the fundamental rules, thus relieving the monotony of the abstract operations, and illustrating their use.

The principles are gradually developed, and explained in a manner calculated to lead the pupil to a full understanding of the difficulties of the science, before he is aware of their existence.

The rules are deduced from a careful analysis of practical problems involving the principles in question—a feature so extensively approved in the author's Series of Arithmetics.

The arrangement of subjects is consecutive and logical, their relation and mutual dependence being pointed out by frequent references.

The examples are numerous, and have been selected with a view to illustrate and familiarize the principles of the Science; while puzzles, calculated to waste the time and energy of the pupil, have been excluded.

Special attention has also been given to Factoring, Generalization, and the application of Algebra to business Formulas.

In these and other respects, it is believed, some advance is made beyond other books of the kind. While adapted to beginners, it covers as much ground as the majority of students master in their Mathematical course.

In presenting this book to the public, the author ventures to hope it may receive the approval so generously bestowed upon his former publications.

J. B. THOMSON.

BROOKLYN, N. Y., *Sept.*, 1877.

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## N O T E .

At the suggestion of several teachers, an Appendix has been added to the present edition of the Practical Algebra, containing a selection of College Examination Problems used for admission to Yale, Harvard, and other colleges. These are preceded by a collection of examples of a similar character, calculated to make experts in Algebra.

# CONTENTS.

---

	PAGE
<b><i>Introduction,</i></b> - - - - -	9
Definitions, - - - - -	9
Algebraic Notation, - - - - -	9
Algebraic Operations, - - - - -	14
Classification of Algebraic Quantities, - - - - -	19
Force of the Signs, - - - - -	21
Axioms, - - - - -	22
<b><i>Addition,</i></b> - - - - -	23
<b><i>Subtraction,</i></b> - - - - -	29
Applications of the Parenthesis, - - - - -	33
<b><i>Multiplication,</i></b> - - - - -	35
Demonstration of the Rule for Signs, - - - - -	37
Multiplying Powers of the Same Letter, - - - - -	38
Principles and Formulas in Multiplication, - - - - -	42
Problems, - - - - -	44
<b><i>Division,</i></b> - - - - -	46
Cancelling a Factor, - - - - -	46
Signs of the Quotient, - - - - -	47
Dividing Powers of the Same Letter, - - - - -	48
Dividing Polynomials, - - - - -	50
Problems, - - - - -	51
<b><i>Factoring,</i></b> - - - - -	53
Prime Factors of Monomials, - - - - -	54
Greatest Common Divisor of Polynomials, - - - - -	63
Demonstration, - - - - -	63
Least Common Multiple of Polynomials, - - - - -	69

	PAGE
<b>Fractions,</b> - - - - -	70
Signs of Fractions, - - - - -	71
Reduction of Fractions, - - - - -	73
Common Denominators, - - - - -	77
Least Common Denominator, - - - - -	79
Addition of Fractions, - - - - -	80
Subtraction of Fractions, - - - - -	82
Multiplication of Fractions, - - - - -	84
Division of Fractions, - - - - -	89
<b>Simple Equations,</b> - - - - -	95
Transposition, - - - - -	96
Reduction of Equations, - - - - -	97
<b>Simultaneous Equations,</b> - - - - -	112
Elimination by Comparison, - - - - -	113
Elimination by Substitution, - - - - -	114
Elimination by Addition or Subtraction, - - - - -	115
Three or More Unknown Quantities, - - - - -	120
<b>Generalization,</b> - - - - -	124
Formation of Rules, - - - - -	126
Generalizing Problems in Percentage, - - - - -	128
Generalizing Problems in Interest, - - - - -	131
Conjunction of the Hands of a Clock, - - - - -	133
<b>Involution,</b> - - - - -	134
Reciprocal Powers, - - - - -	135
Negative Exponents, - - - - -	135
Zero Power, - - - - -	136
<b>Formation of Powers,</b> - - - - -	136
Formation of Binomial Squares, - - - - -	139
Binomial Theorem, - - - - -	140
General Rule, - - - - -	141
Addition and Subtraction of Powers, - - - - -	144
Multiplication and Division of Powers, - - - - -	145
Changing Sign of Exponent, - - - - -	146
<b>Evolution,</b> - - - - -	147
Decimal Exponents, - - - - -	149

	PAGE
Signs of Roots, - - - - -	150
Square Root of the Square of a Binomial, - -	151
Square Root of a Polynomial, - - - - -	152
General Rule, - - - - -	153
<b>Radical Quantities,</b> - - - - -	154
Reduction of Radicals, - - - - -	155
Addition of Radicals, - - - - -	159
Subtraction of Radicals, - - - - -	160
Multiplication of Radicals, - - - - -	161
Division of Radicals, - - - - -	163
Involution of Radicals, - - - - -	164
Evolution of Radicals, - - - - -	165
Changing Radicals to Rational Quantities, - -	166
Radical Equations, - - - - -	169
<b>Quadratic Equations,</b> - - - - -	171
Pure Quadratics. - - - - -	172
Affected Quadratics, - - - - -	175
First Method of Completing a Square, - - -	176
Second Method of Completing a Square, - -	179
Third Method of Completing a Square, - - -	180
Problems, - - - - -	184
Simultaneous Quadratics, - - - - -	187
<b>Ratio,</b> - - - - -	192
<b>Proportion,</b> - - - - -	196
Theorems, - - - - -	198-203
Problems, - - - - -	204
<b>Arithmetical Progression,</b> - - - - -	205
The Last Term of an Arithmetical Series, - -	207
The Sum of an Arithmetical Series, - - -	208
Miscellaneous Formulas in Arith. Progression, -	211
Inserting Arithmetical Means, - - - - -	212
Problems, - - - - -	212
<b>Geometrical Progression,</b> - - - - -	215
The Last Term of a Geometrical Series, - -	216
The Sum of a Geometrical Series, - - - -	217

Miscellaneous Formulas in Geometrical Progression, -	220
Inserting Geometrical Means, - - - -	221
Problems, - - - - -	221
Harmonical Progression, - - - -	223
Infinite Series, - - - - -	226
<b>Logarithms, - - - - -</b>	<b>232</b>
Finding the Logarithm of a Number, - -	235
To find the Number belonging to a Logarithm, - -	236
Multiplication by Logarithms, - - - -	237
Division by Logarithms, - - - -	238
Involution by Logarithms, - - - -	238
Evolution by Logarithms, - - - -	239
Compound Interest by Logarithms, - - -	240
Table of Logarithms, - - - - -	241
<b>Mathematical Induction, - - - -</b>	<b>243</b>
<b>Business Formulas, - - - - -</b>	<b>245</b>
Formulas for Profit and Loss, - - - -	245
Formulas for Simple Interest, - - - -	247
Formulas for Compound Interest, - - - -	248
Formulas for Discount, - - - - -	250
Formulas for Compound Discount, - - - -	251
Formulas for Commercial Discount, - - - -	252
Formulas for Investments, - - - - -	253
Formulas for Sinking Funds, - - - - -	255
Formulas for Annuities, - - - - -	257
<b>Discussion of Problems, - - - - -</b>	<b>261</b>
Problem of the Couriers, - - - - -	262
Imaginary Quantities, - - - - -	265
Indeterminate and Impossible Problems, - -	267
Negative Solutions, - - - - -	268
Horner's Method of Approximation, - - -	269
Test Examples for Review, - - - - -	274
Appendix, - - - - -	283
Collegiate Examination Problems, - - - -	291
Answers, - - - - -	295

# A L G E B R A.

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## CHAPTER I.

### INTRODUCTION.

**Art. 1.** *Algebra*\* is the art of computing by letters and signs. These letters and signs are called *Symbols*.

**2.** *Quantity* is anything which can be measured; as distance, weight, time, number, &c.

**3.** A *Measure* of a quantity is a *unit* of that quantity established by law or custom, as the *Standard Unit*.

Thus, the measure of distance is the *yard*; of weight, the *Troy pound*; of time, the *mean solar day*, etc. *U.S.A. 1850, p. 5.*

### NOTATION.

**4.** *Quantities* in Algebra are expressed by *letters*, or by a combination of *letters and figures*; as, *a*, *b*, *c*, *3x*, *4y*, *5z*, etc.

The first letters of the alphabet are used to express *known* quantities; the last letters, those which are *unknown*.

---

QUESTIONS.—1. What is algebra? Letters and signs called? 2. Quantity? 3. A measure? (4. How are quantities expressed?

\* From the Arabic *al* and *gabron*, reduction of parts to a whole.

5. The *Letters* employed have no *fixed numerical value* of themselves. *Any letter* may represent any number, and the same letter may represent *different* numbers, subject to one limitation; the *same* letter must always stand for the *same* number throughout the *same* problem.

6. The *Relations* of quantities, and the operations to be performed, are expressed by the same *signs* as in Arithmetic.

7. The *Sign of Addition* is a perpendicular cross, called *plus*; \* as, +.

Thus,  $a + b$  denotes the sum of  $a$  and  $b$ , and is read, " $a$  plus  $b$ ," or " $a$  added to  $b$ ."

8. The *Sign of Subtraction* is a short, horizontal line, called *minus*; † as, —.

Thus,  $a - b$  shows that the quantity *after* the sign is to be subtracted from the one *before* it, and is read, " $a$  minus  $b$ ," or " $a$  less  $b$ ."

9. The *Sign of Multiplication* is an oblique cross; as,  $\times$ .

Thus,  $a \times b$  shows that  $a$  and  $b$  are to be multiplied together, and is read, " $a$  times  $b$ ," " $a$  into  $b$ ," or " $a$  multiplied by  $b$ ."

10. *Multiplication* is also denoted by a period between the factors; as,  $a \cdot b$ .

But the *multiplication of letters* is more commonly expressed by writing them together, the signs being omitted.

Thus,  $5abc$  is equivalent to  $5 \times a \times b \times c$ .

11. The *Sign of Division* is a short, horizontal line between the points of a colon; as,  $\div$ .

Thus,  $a \div b$  shows that the quantity before the sign is to be divided by the one after it, and is read, " $a$  divided by  $b$ ."  $\leftarrow$

5. Value of the letters? 6. Relations of quantities expressed? 7. Describe the sign of addition. 8. Subtraction. 9. Multiplication. 10. How else denoted?

\* The Latin term *plus*, signifies *more*.

† The Latin *minus*, signifies *less*.

**12. Division** is also denoted by *writing the divisor under the dividend*, with a short line between them.

Thus,  $\frac{a}{b}$  shows that  $a$  is to be divided by  $b$ , and is equivalent to  $a \div b$ .

**13. The Sign of Equality** is two short, horizontal lines, equal and parallel; as,  $=$ .

Thus,  $a = b$  shows that the quantity before the sign is equal to the quantity after it, and is read, " $a$  equals  $b$ ," or " $a$  is equal to  $b$ ."

**14. The Sign of Inequality** is an acute angle, with the opening turned toward the greater quantity; as,  $>$  and  $<$ .

Thus,  $a > b$  shows that  $a$  is greater than  $b$ , and  $a < b$  shows that  $a$  is less than  $b$ .

**15. The Parenthesis ( ), or Vinculum —**, indicates that the included quantities are taken *collectively*, or as *one* quantity.

Thus,  $3(a + b)$  and  $\overline{a + b} \times 3$ , each denote that the sum of  $a$  and  $b$  is multiplied by 3.

**16. The Double or Ambiguous Sign** is a combination of the signs *plus* and *minus*; as,  $\pm$ .

Thus,  $a \pm b$  shows that  $b$  is to be added to or subtracted from  $a$ , and is read, " $a$  plus or minus  $b$ ."

**17. The character  $\therefore$** , denotes *hence, therefore*.

**18. Every quantity** is supposed to be preceded by the sign *plus* or *minus*. When no sign is prefixed, the sign  $+$  is always understood.

**19. Like Signs** are those which are all *plus*, or all *minus*; as,  $+a + b + c$ , or  $-x - y - z$ .

**20. Unlike Signs** include both *plus* and *minus*; as,  $a - b + c$  and  $-x + y - z$ .

11. Describe the sign of division? 12. How else denoted? 13. The sign of equality? 14. Of inequality? 15. Use of a parenthesis or vinculum? 16. Double sign? 17. Sign for "hence," etc.? 18. By what is every quantity preceded? When none is expressed, what is understood? 19. Like signs? 20. Unlike?

**21. A Coefficient\*** is a number or letter prefixed to a quantity, to show *how many times* the quantity is to be taken. Hence, a coefficient is a *multiplier* or *factor*.

Coefficients may be *numeral*, *literal*, or *mixed*.

Thus, in  $5a$ , 5 is a numeral coefficient of  $a$ ; in  $bc$ ,  $b$  is a literal coefficient of  $c$ ; in  $3dx$ ,  $3d$  is a mixed coefficient of  $x$ .

When *no numeral* coefficient is expressed, 1 is always understood.

Thus,  $xy$  means  $1xy$ .

## EXERCISES IN NOTATION.

### 22. To express a Statement by Algebraic Symbols.

It is required to express the following statement in algebraic symbols:

1. The product of  $a$ ,  $b$ , and  $c$ , divided by the sum of  $c$  and  $d$ , is equal to the difference of  $x$  and  $y$ , increased by the product of  $a$  multiplied by 7.

$$\text{Ans. } a \times b \times c \div (c + d) = (x - y) + 7a.$$

$$\text{Or } \frac{abc}{c + d} = (x - y) + 7a. \text{ Hence, the}$$

**RULE.**—For the words, substitute the signs which indicate the relations of the quantities and the operations to be performed.

Express the following by algebraic symbols:

2. The sum of  $4c$ ,  $d$ , and  $m$ , diminished by  $5x$ , equals the product of  $a$  and  $b$ .

3. The product of  $5c$  and  $d$ , increased by the quotient of  $a$  divided by  $b$ , equals the product of  $x$  and  $y$ .

21. A coefficient? When no coefficient is expressed, what is understood?

22. How translate a statement from common language into algebraic symbols?

\* Coefficient, Latin, *con*, with, and *efficere*, to effect; literally, a co-operator.

4. The quotient of  $3b$  divided by  $5c$ , increased by  $4m$ , equals the sum of  $c$  and  $6d$ , diminished by the product of  $7a$  and  $x$ .

5. If to the difference between  $a$  and  $b$ , we add the product of  $x$  into  $y$ , the sum will be equal to  $m$  multiplied by  $6n$ .

6. The difference between  $x$  and  $y$ , added to the sum of  $4a$  and  $b$  minus  $m$ , equals the product of  $c$  and  $d$ , increased by 15 times  $m$ .

~~13~~ These and the following exercises should be supplemented by dictation, until the learner becomes familiar with them.

### 23. To translate *Algebraic Expressions* into Common Language.

Express the following statement in common language:

$$1. \frac{a+b}{d} = 2abc - (x+y) + \frac{d}{ab}.$$

Substituting words for signs, we have the sum of  $a$  and  $b$ , divided by  $d$ , equals twice the product of  $a$ ,  $b$ , and  $c$ , diminished by the sum of  $x$  and  $y$ , increased by the quotient of  $d$  divided by the product of  $a$  and  $b$ , *Ans.* Hence, the

**RULE.**—*For the signs indicating the given relations and operations, substitute words.*

Express the following in common language:

$$2. \frac{2ab}{x} + a - b = \frac{a+b}{c} + axy - 4cd.$$

$$3. \frac{3b+c}{8} + 3x = \frac{3cd}{a} + xyz - \frac{c}{d}.$$

$$4. \frac{3a}{5} - ax + bc = \frac{4a-b}{x} + \frac{cd}{4} - 3x.$$

$$5. \frac{abc-x}{3d} \pm 3x + 5y = \frac{cdh+x}{2a} - xy.$$

$$6. \frac{4axy}{5a} + \frac{a-b}{x} = \frac{x+y}{a} - \frac{2a+d}{3c}.$$

## ALGEBRAIC OPERATIONS.

**24.** An *Algebraic Operation* is combining quantities according to the principles of algebra.

**25.** A *Theorem* is a statement of a principle to be proved.

**25.a.** A *Problem* is something proposed to be done, as a question to be solved.

**26.** The *Equality* between two quantities is denoted by the sign  $=$ . (Art. 13.)

**27.** The *Expression of Equality* between two quantities is called an *Equation*. Thus,  $15 - 3 = 7 + 5$  is an equation.

## PROBLEMS.

**28.** The following problems are solved by combining the preceding principles with those of Arithmetic.

1. A and B found a purse containing 12 dollars, and divided it in such a manner that B's share was three times as much as A's. How many dollars did each have?

BY ARITHMETIC.—A had 1 share and B 3 shares; now 1 share + 3 shares are 4 shares, which are equal to 12 dollars. If 4 shares equal 12 dollars, 1 share is equal to as many dollars as 4 is contained times in 12, which is 3. Therefore, A had 3 dollars, and B had 3 times as much, or 9 dollars.

BY ALGEBRA.—We represent A's share by  $x$ , and form an equation by treating this letter as we treat the answer in proving an operation. If  $x$  represent A's share,  $3x$  will represent B's, and  $x + 3x = 12$  dollars, the sum of both. Uniting the terms, we have the equation,  $4x = 12$  dollars. To remove the coefficient 4, we

## OPERATION.

Let  $x =$  A's share,  
then  $3x =$  B's share,  
and  $x + 3x = 12$  dollars,  
that is,  $4x = 12$  dollars.  
Hence,  $x = 3$  dol., A.  
 $3x = 9$  dol., B.

24. What is an algebraic operation? 25. A problem? A solution? 26. Equality denoted? 27. The expression of equality called?

divide both sides of the equation by it. For, if equals are divided by equals, the quotients are equal. Therefore,  $x = 3$  dollars, A's share, and  $3x = 9$  dollars, B's share. (Ax. 5.)

PROOF.—By the first condition, 9 dollars, B's share = 3 times 3 dollars, A's share. By the second condition, 9 dollars + 3 dollars = 12 dollars, the sum found. Hence,

**29.** When a quantity on either side of the equation has a coefficient, that coefficient may be removed, *by dividing every term on both sides of the equation by it.*

2. A and B together have 15 pears, and A has twice as many as B: how many has each?

By ALGEBRA.—If  $x$  represents B's number,  $2x$  will represent A's, and  $x + 2x$ , or  $3x$ , will represent the number of both. Dividing both sides by the coefficient 3, we have  $x = 5$  pears, B's number, and  $2x = 10$  pears, A's.

OPERATION.

Let  $x =$  B's number;  
then  $2x =$  A's "  
and  $3x = 15$  pears.  
 $\therefore x = 5$  pears, B's.  
 $2x = 10$  pears, A's.

NOTE.—It is advisable for the learner to solve each of the following problems by Arithmetic and by Algebra.

3. A lad bought an apple and an orange for 8 cents, paying 3 times as much for the orange as for the apple. What was the price of each?

4. A farmer sold a cow and a ton of hay for 40 dollars, the cow being worth 4 times as much as the hay. What was the value of each?

5. The sum of two numbers is 36, one of which is 3 times the other. What are the numbers?

6. A, B, and C have 28 peaches; B has twice as many as C, and A twice as many as B. How many has each?

7. A father is 3 times the age of his son, and the sum of their ages is 48 years. How old is each?

8. A and B trade in company, and gain 100 dollars. If A puts in 4 times as much as B, what will be the gain of each?

9. The sum of three numbers is 90. The second is twice the first, and the third as many as the first and second: what are the numbers?

10. A cow and calf were sold for 63 dollars, the cow being worth 8 times as much as the calf. What was the value of each?

11. A man being asked the price of his horse, replied that his horse, saddle and bridle together were worth 126 dollars; that the saddle was worth twice as much as the bridle, and the horse 7 times as much as both the other. What was each worth?

12. A man bequeathed \$36,000 to his wife, son and daughter, giving the son twice as much as the daughter, and the wife 3 times as much as the son and daughter. What did each receive?

13. The sum of three numbers is 1872: the second is 3 times the first, and the third equals the other two. What are the numbers?

## POWERS AND ROOTS.

**30.** A *Power* is the product of two or more *equal* factors.

Thus, the product  $2 \times 2$ , is the *square* or *second* power of 2;  $x \times x \times x$  is the *cube* or *third* power of  $x$ .

**31.** The *Index* or *Exponent* of a *power* is a figure or letter placed at the right, above the quantity.

Thus,  $a^1$  denotes  $a$ , or the *first* power.

$a^2$  "  $a \times a$ , the *square*, or *second* power.

$a^3$  "  $a \times a \times a$ , the *cube*, or *third* power, etc.

**32.** A *Root* is one of the *equal factors* of a quantity.

**33. Roots** are denoted by the *Radical Sign*  $\sqrt{\quad}$  prefixed to the quantity, or by a *fractional exponent* placed after it.

Thus,  $\sqrt{a}$ ,  $a^{\frac{1}{2}}$ , or  $\sqrt[2]{a}$  denote the square root of the quantity  $a$ ;  $\sqrt[3]{a}$  shows that the cube root of  $a$  is to be extracted, etc.

**34. The Index of the Root** is the figure placed over the radical sign. The index of the square root is usually omitted.

(For negative indices, see Arts. 256, 258.)

Read the following examples:

- |                       |   |
|-----------------------|---|
| 1. $a^2 + 3a$ .       | 7. $4(a - b)^2$ .                           |
| 2. $b^3 - c^2$ .      | 8. $a^2 + 2ab + b^2$ .                      |
| 3. $a + b^2 - c$ .    | 9. $\sqrt{a + b}$ .                         |
| 4. $x^3 - y + y^2$ .  | 10. $\sqrt[3]{a^2 - b^2}$ .                 |
| 5. $2y^2 + z^3 - z$ . | 11. $2a^{\frac{1}{2}} + c^{\frac{1}{2}}$ .  |
| 6. $3(a^2 + b)$ .     | 12. $4x^{\frac{1}{2}} + 2y^{\frac{1}{2}}$ . |

Write the following in algebraic language:

13. The square of  $a$  plus the square of  $b$ .
14. The square of the sum of  $a$  and  $b$ .
15. The sum of  $a$  and  $b$ , minus the square of  $c$ .
16. The square root of  $a$ , plus the square root of  $x$ .
17. The cube root of  $x$ , minus the fifth power of  $y$ .
18. The cube root of  $a$ , plus the square of  $b$ .

## ALGEBRAIC EXPRESSIONS.

**35. An Algebraic Expression** is any quantity expressed in algebraic language; as,  $3a$ ,  $5a - 7b$ , etc.

**36. The Terms** of an algebraic expression are those parts which are connected by the signs  $+$  and  $-$ .

Thus, in  $a + b$ , there are two terms; in  $x + y \times z - a$  there are three.

---

32. A root? 33. How denoted? 34. What is the figure placed over the radical sign called? 35. What is an algebraic expression? 36. Its terms?

**NOTE.**—Letters combined by the signs  $\times$  or  $+$  do not constitute separate terms. Such a combination, to form a term, must have the sign  $+$  or  $-$  prefixed to it, and the operations indicated by the signs  $\times$  or  $+$  must be performed before the terms can be added to or subtracted from the preceding term. (Art. 36.) Thus,  $a+b \times c$  has two terms,  $b \times c$  forming one term and  $a$  the other.

**37.** The *Dimensions* of a term are its several *literal* factors.

**38.** The *Degree* of a term depends on the number of its *literal factors*, and is always equal to the *sum* of their exponents.

Thus,  $ab$  contains *two factors*,  $a$  and  $b$ , and is of the *second degree*.  $a^2x$  contains *three factors*,  $a$ ,  $a$ , and  $x$ , and is of the *third degree*.  $b^2x^3$  contains *five factors*,  $b$ ,  $b$ ,  $x$ ,  $x$ ,  $x$ , and is of the *fifth degree*.

**39.** The *Numerical Value* of an algebraic expression is the *number* which it represents when its terms are combined as indicated by the signs. (Art. 36.)

**40.** To Find the *Numerical Value* of an algebraic expression.

1. If  $a = 5$ ,  $b = 7$ , and  $x = 9$ , what is the value of  $6a + 8b + 3x$ ?

**ANALYSIS.**—Since  $a = 5$ ,  $6a$  must equal 6 times 5, or 30; since  $b = 7$ ,  $8b$  must equal 8 times 7, or 56; and since  $x = 9$ ,  $3x$  must equal 3  $\times$  9, or 27. Now  $30 + 56 + 27 = 113$ . Therefore, the value of the given expression is 113. Hence, the

**OPERATION.**

$$6a = 5 \times 6 = 30$$

$$8b = 7 \times 8 = 56$$

$$3x = 3 \times 9 = 27$$

*Ans.* 113

**RULE.**—For the letters, substitute the figures which the letters represent, and perform the operations indicated by the signs.

2. If  $b = 3$ ,  $c = 5$ , and  $d = 8$ , what is the value of  $5b + 7c + 6d$ ?

**SUGGESTION.**  $15 + 35 + 48 = 98$ , *Ans.*

---

37. The dimensions of a term? 38. Degree? 39. Numerical value of an algebraic expression? 40. How found?

Find the numerical value of the following expressions, when  $a = 2$ ,  $b = 3$ ,  $c = 4$ ,  $d = 5$ , and  $x = 6$ .

3.  $4a + 6ab + 5c =$  how many? *Ans.* 64.

4.  $(a + b) \times c \times d \div (x \div c) =$  how many?

5.  $(x - a) + ax + (c \div a) =$  how many?

6.  $x \div 2 + (d - c) + bc - x =$  how many?

7.  $dx - (a \times c) + (a \times b) + x =$  how many?

8.  $d + (x \times a) + a - x + c =$  how many?

## CLASSIFICATION OF ALGEBRAIC QUANTITIES.

**41.** Quantities in Algebra are primarily divided into *known* and *unknown*.

**42.** A *Known Quantity* is one whose value is given.

An *Unknown Quantity* is one whose value is not given.

These quantities are subdivided into like and unlike, positive and negative, simple, compound, monomials, etc.

**43.** *Like Quantities* are those which are expressed by the *same power of the same letters*; as,  $a$  and  $2a$ ,  $2x^2$  and  $x^2$ .

**44.** *Unlike Quantities* are those which are expressed by different letters, or by different powers of the *same letters*; as  $2x$  and  $3y$ ,  $2x$  and  $x^2$ .

NOTE.—An exception must be made in cases where letters are regarded as coefficients. Thus,  $ax^2$  and  $bx^2$  are like quantities, when  $a$  and  $b$  are considered coefficients.

**45.** A *Positive Quantity* is one that is to be *added*, and has the sign  $+$  prefixed to it; as,  $4a + 3b$ .

**46.** A *Negative Quantity* is one that is to be *subtracted*, and has the sign  $-$  prefixed to it; as,  $4a - 3b$ .

---

41. How are quantities in Algebra primarily divided? 42. A known quantity? Unknown? 43. Like quantities? 44. Unlike? 45. A positive quantity? 46. A negative?

**47.** The terms *Positive* and *Negative* denote oppositeness of direction in the use of the quantities to which they are applied. If lines running *North* from any point are *positive*, those running *South* are *negative*. If *future time* is *positive*, *past time* is *negative*; if *credits* are *positive*, *debts* are *negative*, etc.

**48.** A *Simple Quantity* is a single letter, or several letters written together without the sign  $+$  or  $-$ ; as,  $a$ ,  $ab$ ,  $3xy$ .

**49.** A *Compound Quantity* is two or more simple quantities connected by the sign  $+$  or  $-$ ; as  $3a + 4b$ ,  $2x - y$ .

**50.** A *Monomial*\* has but one term; as,  $a$ ,  $3b$ .

**51.** A *Binomial*† has two terms; as,  $a + b$ ,  $a - b$ .

NOTES.—I. The expression  $a - b$  is often called a *residual*, because it denotes that which remains after a part is subtracted.

2. A *binomial* is sometimes called a *polynomial*.

**52.** A *Trinomial*‡ has three terms; as,  $a + b - c$ .

**53.** A *Polynomial*|| has two or more terms; as,  $a + b - c + x$ .

**54.** An *Homogeneous Polynomial* has all its terms of the same degree.

Thus,  $2ab + cd + 5xy$  is homogeneous; but  $4abc + c^2 + 5x$  is not.

**55.** The *Reciprocal* of a quantity is a unit divided by that quantity.

Thus, the reciprocal of  $a$  is  $\frac{1}{a}$ ; the reciprocal of  $a + b$  is  $\frac{1}{a + b}$ .

47. What do the terms positive and negative denote? 48. A simple quantity? 49. A compound? 50. A monomial? 51. A binomial? Note. The expression  $a - b$  called? 52. A trinomial? 53. A polynomial? 54. When homogeneous? 55. The reciprocal of a quantity?

\* Greek, *monos*, single, and *nomē*, term, having *one term*.

† Latin, *bis*, two, and *nomē*, name (a hybrid), *two terms*.

‡ Greek, *treis*, three, and *nomē*, name, having *three terms*.

|| Greek, *polus*, many, and *nomē*, name, having *many terms*.

## FORCE OF THE SIGNS.

**56.** *Each term* of an algebraic expression is preceded by the sign  $+$  or  $-$ , expressed or understood. (Art. 18.)

The *Force* of each of these signs is limited to the *term* which follows it; as,  $7 + 5 - 3 = 12 - 3 = 9$ ;  $15 - 6 + 8 = 9 + 8 = 17$ .

**57.** If a term, preceded by the sign  $+$  or  $-$ , is combined with other letters by the sign  $\times$  or  $\div$ , each of these letters forms a part of that term, and the operations indicated, taken in their order, must be performed before any part of the term can be *added* to or *subtracted* from any other term.

Thus, the expression  $12 + 4 \times 2$ , shows that 4 is to be multiplied by 2 and the product added to 12; and is equal to 20.

In like manner, the expression  $16 - 8 \div 2$ , shows that 8 is to be divided by 2 and the quotient subtracted from 16, and is equal to 12.

**58.** If two or more terms joined by  $+$  or  $-$  are to be subjected to the *same operation*, they must be connected by a *parenthesis* or *vinculum*.

Thus, if  $a + b$  or  $a - b$  is to be multiplied or divided by  $c$ , the operations are indicated by  $(a + b) \times c$ , or  $c(a + b)$ ;  $(a - b) \div c$ , or  $\frac{a - b}{c}$ .

## EXERCISES.

1.  $50 + 5 \times 2 =$  what number?
2.  $50 - 5 \times 2 =$  what number?
3.  $ac + 4b \times 2 =$  what?
4.  $5b - 6d \div 3 =$  what?
5.  $15 + 5 \times 3 + 10 \div 2 =$  what?
6.  $(18 - (2 \times 4 \div 2)) + 10 =$  what?
7.  $(3x + 8y \div 4) + a \times b =$  what?
8.  $6b - (7c \times x) + 9a \div 3 =$  what?
9.  $(b + c) \times xy =$  what?

---

56. By what are all algebraic terms preceded? The force of each of these signs?  
 57. Of the signs  $\times$  and  $\div$ ? 58. Of the parenthesis and vinculum?

10.  $3x \times 5y \div 2z + a = \text{what?}$

11.  $(b - a) \div xy + 2z = \text{what?}$

12.  $3x + xy + 2z \times 3y = \text{what?}$

13. The difference of  $x$  and  $y$ , multiplied by  $a$  less  $b$ , and the product divided by  $d = \text{what?}$

Find the value of the following expressions, in which  $a = 3$ ,  $b = 4$ ,  $c = 2$ ,  $x = 6$ ,  $y = 8$ , and  $z = 10$ :

14.  $a + (a \times x) \div c + y \times z = \text{what?}$

15.  $2b \div (x - b) + a \times b \times y + 2z = \text{what?}$

### AXIOMS.

59. An *Axiom* is a self-evident truth.

1. Things which are equal to the *same* thing, are equal to each other.

2. If equals are *added* to equals, the *sums* are equal.

3. If equals are *subtracted* from equals, the *remainders* are equal.

4. If equals are *multiplied* by equals, the *products* are equal.

5. If equals are *divided* by equals, the *quotients* are equal.

6. If a quantity is *multiplied* and *divided* by the same quantity, its *value* is not altered.

7. If the same quantity is *added to* and *subtracted from* another quantity, the *value* of the latter is not altered.

8. The *whole* is greater than its *part*.

9. The *whole* is equal to the sum of *all its parts*.

10. Like *powers* and like *roots* of equal quantities, are *equal*.

NOTE.—The importance of thoroughly understanding the *definitions* and *principles* cannot be too deeply impressed upon the mind of the learner. The questions at the foot of the page are designed to direct his attention to the more important points. Teachers, of course, will not be confined to them.

## CHAPTER II.

### ADDITION.

**60.** *Addition* in Algebra is uniting two or more quantities and *reducing* them to the simplest form.

**61.** The *Result* is called the *Sum* or *Amount*.

**62.** *Quantities* expressed by *letters* are regarded as *concrete* quantities. Hence, their coefficients may be added, subtracted, multiplied, and divided like concrete numbers.

Thus,  $3a$  and  $4a$  are  $7a$ ,  $4b$  and  $5b$  are  $9b$ , as truly as 3 apples and 4 apples are 7 apples, or as 4 bushels and 5 bushels are 9 bushels.

### PRINCIPLES.\*

**63.** 1°. *Like quantities only can be united in one term.*

2°. *The sum of two or more quantities is the same in whatever order they are added.*

### CASE I.

**64.** To Add like Monomials which have like signs.

1. What is the sum of  $15ab + 13ab + 19ab$ ?

**ANALYSIS.**—These terms are like quantities and have like signs. (Art. 19.) We therefore add the coefficients, to the sum annex the common letters, and prefix the common sign. The result,  $+ 47ab$ , is the answer required. (Ax. 9.)

**OPERATION.**

$$\begin{array}{r} + 15ab \\ + 13ab \\ + 19ab \\ \hline + 47ab, \text{ Ans.} \end{array}$$

---

60. What is addition? 61. The result called? 62. How are quantities expressed by letters regarded? 63. First principle? Second?

\* The expressions 1°, 2°, 3°, etc., denote *first, second, third, etc.*

2. What is the sum of  $-14xy$ ,  $-16xy$ , and  $-18xy$ ?

ANALYSIS.—Since these terms are like quantities, and have like signs, we add them as before, and prefix the sign  $-$  to the result, for the reason that all the quantities have the sign  $-$ . Hence, the

$$\begin{array}{r} -14xy \\ -16xy \\ -18xy \\ \hline -48xy, \text{ Ans.} \end{array}$$

RULE.—Add the coefficients; to the sum annex the common letters, and prefix the common sign.

(3.)	(4.)	(5.)	(6.)	(7.)
$3ab$	$5xy$	$7a^2$	$-7bcd$	$-4x^2y^2$
$5ab$	$8xy$	$3a^2$	$-3bcd$	$-3x^2y^2$
$6ab$	$xy$	$4a^2$	$-5bcd$	$-x^2y^2$
<u><math>7ab</math></u>	<u><math>3xy</math></u>	<u><math>a^2</math></u>	<u><math>-8bcd</math></u>	<u><math>-8x^2y^2</math></u>

8. Add  $5ab^2 + 17ab^2 + 23ab^2$ .

9. Add  $-8abx^2y^2 - 3abx^2y^2 - 28abx^2y^2$ .

10. Add  $5b^2dm^3 + 7b^2dm^3 + 9b^2dm^3 + 8b^2dm^3$ .

11. If  $3a + 5a + a + 7a = 48$ , to what is  $a$  equal?

SOLUTION.  $3a + 5a + a + 7a = 16a$ ; hence,  $a = 48 \div 16$ , or 3. Ans.

12. If  $4bc + 9bc + 2bc + 5bc = 80$ , to what is  $bc$  equal?

13. If  $xy + 3xy + 5xy + 4xy = 65$ , to what is  $xy$  equal?

## CASE II.

65. To Add like Monomials which have *Unlike* signs.

14. What is the sum of  $5ab - 3ab - 7ab + 9ab + 6ab - 8ab$ ?

ANALYSIS.—For convenience in adding, we write the *negative* terms one under another in the right-hand column, with the sign  $-$  before each, and the *positive* terms in the next column on the left.

OPERATION.

$$\begin{array}{r} 5ab - 3ab \\ 9ab - 7ab \\ 6ab - 8ab \\ \hline 20ab - 18ab = 2ab, \text{ Ans.} \end{array}$$

We then find the sum of the coefficients of the positive and negative

terms separately ; and taking the less sum from the greater, the result  $2ab$ , is the answer. Hence, the

**RULE.—I.** Write the positive and negative terms in separate columns with their proper signs, and find the sum of the coefficients of each column separately.

**II.** From the greater subtract the less ; to the remainder prefix the sign of the greater, and annex the common letters.

**NOTE.**—If two equal quantities have *opposite* signs, they balance each other, and may be omitted.

15. Add  $4d + 3d - 5d + 6d - 2d$ . Ans.  $6d$ .

16. Add  $-5x + 6x + 8x - 3x + 9x - 7x$ .

17. Add  $3abc + 12abc - 6abc + 5abc - 10abc - 3abc$ .

18. Add  $2b - 5b + 4b - 6b - 7b$ .

19. Add  $-6y + 4y - 8y - 9y + 8y - y$ .

20. Add  $4m + 16m - 8m - 9m + 5m - 10m$ .

21. If  $6ab + 14ab - 7ab + 15ab - 12ab + 16ab = 32$ , to what is  $ab$  equal?

22. To what is  $bcd$  equal, if  $bcd - 3bcd + 4bcd + 4bcd - 5bcd = 75$ ?

**REMARK.**—The *sum* in Arithmetic is *always* greater than any of its parts. But, in Algebra, it will be observed, the *sum* of a *positive* and *negative* quantity is *always less* than the *positive* quantity. It is thence called *Algebraic Sum*.

**66. Unlike Quantities** cannot be united in *one term*. Their *sum* is indicated by writing them one after another, with their proper signs. (Art. 63, Prin. 1.)

Thus, the sum of  $7g$  and  $3d$  is neither  $10g$  nor  $10d$ , any more than 7 guineas and 3 dollars are 10 guineas or 10 dollars. Their sum is  $7g + 3d$ . (Art. 63, Prin. 1.)

**67. Polynomials** are added by uniting like quantities, as in adding monomials.

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65. How add monomials having unlike signs? *Rem.* What is true of the sum in Arithmetic? In Algebra? 66. How add unlike quantities? 67. Polynomials?

23. What is the sum of the polynomial  $3ab - 3b + 4d - 3x$ ;  $-5ab + 4x - c - 2d$ ; and  $bg + d + 2ab + b$ ?

ANALYSIS.—For convenience, we write the quantities so that like terms shall stand one under another, and uniting those which are alike, the result is  $-2b + 3d + x + bg - c$ .

OPERATION.

$$\begin{array}{r}
 3ab - 3b + 4d - 3x - c \\
 - 5ab + \quad b - 2d + 4x \\
 \quad 2ab \qquad + \quad d \qquad + \quad bg \\
 \hline
 - 2b + 3d + x + bg - c, \text{ Ans.}
 \end{array}$$

68. From the preceding illustrations and principles we deduce the following

### GENERAL RULE.

I. Write the given quantities so that like terms shall stand one under another.

II. Unite the terms which are alike, and to the result annex the unlike terms with their proper signs. (Art. 65.)

1. Add  $5a - 3a + 6a + 7a + 9a + 2b - 3d$ .
2. Add  $8mn + 3mn - 4mn + 9mn - xy + bc$ .
3. Add  $3bc - 7bc + xy - mn + 11bc + 9bc$ .
4. Add  $5ab - 3mn - ab + 3ab + 2x - 4ab + ab$ .
5. Add  $3xy - xy + ab - 7xy + b + 8xy - xy + 13xy$ .

69. *Compound Quantities* inclosed in a parenthesis, are taken *collectively*, or as *one* quantity. Hence, if the quantities are *alike*, their *coefficients* and *exponents* are treated as the coefficients and exponents of like monomials. (Art. 64.)

6. What is the sum of  $3(a+b) + 5(a+b) + 7(a+b)$ ?

SOLUTION.  $3(a+b)$  and  $5(a+b)$  and  $7(a+b)$  are  $15(a+b)$ . Ans.

7. Add  $13(a+b) + 15(a+b) - 7(a+b)$ .
8. Add  $8c(x-y) + 7c(x-y) - 5c(x-y) + 9c(x-y)$ .
9. Add  $3a\sqrt{xy} + 5a\sqrt{xy} - 7a\sqrt{xy} + 6a\sqrt{xy}$ .
10. Add  $5\sqrt{a} + 3\sqrt{a} - 8\sqrt{a} + 9\sqrt{a} - 3\sqrt{a}$ .
11. Add  $8\sqrt{x-y} - 3\sqrt{x-y} + 5\sqrt{x-y}$ .

68. The general rule for addition? 69. How add quantities included in a parenthesis?

**70.** The *sum* of *unlike* quantities having a *common* letter or letters, may be expressed by *inclosing the other letters*, with their signs and coefficients, in a parenthesis, and *annexing or prefixing* the *common* letter or letters to the result.

12. What is the sum of  $5ax + 3bx - 4cx$ ?

SOLUTION.  $5ax + 3bx - 4cx = (5a + 3b - 4c)x$ , or  $x(5a + 3b - 4c)$ . *Ans.*

13. Add  $7a - 6ba + 3da - 3ma$ .

14. Add  $aby + 3y - 2cy - 5my$ .

15. Add  $9m + abm - 7cm + 3dm$ .

16. Add  $13ax - 3bx + cx - 3dx + mx$ .

17. Add  $axy + bxy - cxy$ .

PROBLEMS.

**71.** Problems requiring equal quantities to be added to each side of the equation.

1. A has 3 times as many marbles as B, lacking 6; and both together have 58. How many has each?

ANALYSIS.—If  $x$  represents B's number, then will  $3x - 6$  represent A's, and  $x + 3x - 6 = 58$ , the sum of both. To remove  $-6$ , we add an *equal positive* quantity to each side of the equation. (Axiom 2.)  
Uniting the terms, we have  $4x = 64$ , and  $x = 16$ , B's, and 3 times  $16 - 6$ , or  $42 =$  A's No.

OPERATION.

Let	$x =$ B's No.;
then	$3x - 6 =$ A's "
	$x + 3x - 6 = 58$ , both.
	$x + 3x - 6 + 6 = 58 + 6$
	$4x = 64$
	$x = 16$ , B's No.
	$3x - 6 = 42$ , A's "

**72.** When a *negative* quantity occurs on either side of an equation, that quantity may be removed by *adding an equal positive quantity to both sides*.

NOTE.—In forming the equation, we treat  $x$  as we do the answer in proving an operation.

2. A kite and a ball together cost 46 cents, and the kite cost 2 cents less than twice the cost of the ball. What was the cost of each?

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70. How may the sum of unlike quantities which have a common letter be expressed?

3. In a basket there are 75 peaches and pears ; the number of pears being double that of the peaches, wanting 3. How many are there of each ?

4. The sum of two numbers is 85, and the greater is 5 times the less, wanting 5. What are the numbers ?

5. A certain school contains 40 pupils, and there are twice as many girls, lacking 5, as boys. How many are there of each ?

6. If  $44x + 65x - 24 = 85$ , what is the value of  $x$  ?

7. If  $7x - 3 + 2x = 60$ , what is the value of  $x$  ?

8. If  $4y + 2y + 5y - 7 = 70$ , what is the value of  $y$  ?

9. The whole number of votes cast for A and B at a certain election was 450 ; A had 20 votes less than 4 times the number for B. How many votes had each ?

10. The sum of two numbers is 177 ; the greater is 3 less than 4 times the smaller. What are the numbers ?

11. What is the value of  $y$ , if  $4y + 3y + 2y - 12 = 60$  ?

12. A lad bought a top and a ball for 32 cents ; the price of the ball was 3 times that of the top, minus 4 cents. What was the price of each ?

13. A man being asked the price of his saddle and bridle, replied that both together cost 40 dollars, the former being 4 times the price of the latter, minus 5 dollars. What was the price of each ?

14. A lad spent a dollar during a holiday, using three times as much of it in the afternoon as in the morning, minus 4 cents ; how much did he spend in each part of the day ?

Find the value of  $x$  in the following equations :

15.  $3x + 6x + 4x + 5x - 8 = 154$ .

*Ans.* 9.

16.  $2x + 5x + 3x - 10 = 130$ .

17.  $4x + 3x + 7x - 12 = 86$ .

18.  $10x - 4x + 9x - 25 = 155$ .

19.  $15x - 7x - 2x - 60 = 300$ .

20.  $18x - 4x + x - 75 = 225$ .

## CHAPTER III.

### SUBTRACTION.

**73.** *Subtraction* is finding the *difference* between two quantities.

The *Minuend* is the quantity from which the subtraction is made.

The *Subtrahend* is the quantity to be subtracted.

The *Difference* is the result found by subtraction.

**74.** Since quantities expressed by letters are regarded as *concrete*, the *coefficient* of one letter may be subtracted from that of another, like concrete numbers. (Art. 62.)

Thus,  $7a - 3a = 4a$ ;  $8b - 5b = 3b$ .

### PRINCIPLES.

**75.** 1°. *Like quantities only can be subtracted one from another.*

2°. *The sum of the difference and subtrahend is equal to the minuend.*

3°. *Subtracting a positive quantity is equivalent to adding an equal negative one.*

Thus, let it be required to subtract  $+4$  from  $6+4$ .

Taking  $+4$  from  $6+4$ , leaves 6.

Adding  $-4$  to  $6+4$ , we have  $6+4-4$ .

But (Ax. 7)  $6+4-4$  is equal to 6.

4°. *Subtracting a negative quantity is the same as adding an equal positive one.*

---

73. Define subtraction. The Minuend. Subtrahend. Difference. 75. Name the first principle. Second. Illustrate Prin. 3 upon the blackboard. Illustrate Prin. 4.

Thus, let it be required to subtract  $-4$  from  $10-4$ .

Taking  $-4$  from  $10-4$ , leaves  $10$ .

Adding  $+4$  to  $10-4$ , we have  $10-4+4$ .

But (Ax. 7)  $10-4+4$  is equal to  $10$ .

Again, if the assets of an estate be \$500, and the liabilities \$300, the former being considered *positive* and the latter *negative*, the net value of the estate will be  $\$500 - \$300 = \$200$ . Taking \$50 from the *assets* has the same effect on the *balance* as adding \$50 to the *liabilities*. In like manner, taking \$50 from the *liabilities* has the same effect as adding \$50 to the *assets*.

## 76. To Find the *Difference* between two like Quantities.

This proposition includes three classes of examples, as will be seen in the following illustrations:

### 1. From $25a$ subtract $17a$ .

REMARK.—1. In this example the *signs* are *alike*, and the subtrahend is *less* than the minuend. Subtracting a positive quantity is equivalent to adding an *equal negative* one. (Prin. 3.) We therefore change the sign of the subtrahend, and then unite the terms as in addition. Thus,  $25a - 17a = 8a$ .

OPERATION.

$$\begin{array}{r} 25a \text{ Minuend.} \\ - 17a \text{ Subtrahend.} \\ \hline 8a \text{ Difference.} \end{array}$$

### 2. From $4a$ subtract $7a$ .

REMARK.—2. In this example the *signs* are *alike*, but the subtrahend is *greater* than the minuend. Changing the sign of the subtrahend, and uniting the terms as before, the subtrahend cancels the minuend, and has  $-3a$  left. (Prin. 3.)

OPERATION.

$$\begin{array}{r} 4a \text{ Minuend.} \\ - 7a \text{ Subtrahend.} \\ \hline - 3a \text{ Difference.} \end{array}$$

### 3. From $45ab$ subtract $-29ab$ .

REMARK.—3. In this example the *signs* are *unlike*. Subtracting a *negative* quantity is the same as adding an *equal positive* one. (Prin. 4.) Changing the sign of the subtrahend and proceeding as before, we have  $45ab + 29ab = 74ab$ . *Ans.*

OPERATION.

$$\begin{array}{r} 45ab \text{ Minuend.} \\ + 29ab \text{ Subtrahend.} \\ \hline 74ab, \text{ Ans.} \end{array}$$

4. From  $9bc + 7d - 5x$ , take  $3bc + 2d - 4x$ .

ANALYSIS. — In subtraction of polynomials, for convenience, we place like terms under each other. Then, changing the signs of all the terms in the subtrahend, we unite them as before.

OPERATION.

$$\begin{array}{r} 9bc + 7d - 5x \\ - 3bc - 2d + 4x \\ \hline 6bc + 5d - x, \text{ Ans.} \end{array}$$

77. From the preceding illustrations and principles we deduce the following

### GENERAL RULE.

I. Write the subtrahend under the minuend, placing like terms one under another.

II. Change the signs of all the terms of the subtrahend, or suppose them to be changed, from + to −, or from − to +, and then proceed as in addition. (Art. 75, Prin. 3, 4.)

NOTES.—1. Unlike quantities can be subtracted only by changing the signs of all the terms of the subtrahend, and then writing them after the minuend. (Art. 66.)

2. As soon as the student becomes familiar with the principles of subtraction, instead of actually changing the signs of the subtrahend, he may simply suppose them to be changed.

### EXAMPLES.

1. From  $43c + d$ , take  $25c + d$ . Ans.  $18c$ .
2. From  $49x$ , take  $23x + 3$ . Ans.  $26x - 3$ .
3. From  $28xyz$ , take  $14xyz$ .
4. From  $-43ab$ , take  $+19ab$ .
5. From  $4ab$ , take  $-15ab$ .
6. From  $43xy$ , take  $+16xy$ .

	(7.)	(8.)	(9.)	(10.)
From	$20ac$	$42ax^2$	$37a^2b$	$-29x^2y^2$
Take	$-23ac$	$5ax^2$	$-14a^2b$	$+15x^2y^2$

77. General rule for subtraction? Note. How subtract unlike quantities.

	(11.)	(12.)	(13.)	(14.)
From	$31a^2b$	$19abx^2$	$-33m^2x$	$41x^2y$
Take	$-7a^2b$	$19abx^2$	$+44m^2x$	$-12x^2y$

15. A is worth \$100, and B owes \$50 more than he is worth. What is the difference in their pecuniary standing?

16. What is the difference in temperature, when the thermometer stands 15 degrees *above* zero, and when at 10 degrees below?

17. By speculation, A gained on a certain day \$275, and B lost \$145. What was the difference in the results of their operations?

	(18.)	(19.)	(20.)
From	$7xy - 8a$	$8b^2 + 7am$	$13x^2 - 7y^2$
Take	$3xy - 2a$	$-5b^2 - 9am$	$-5x^2 - 8y^2 - 6a$

21. From  $13ab + d - x$ , subtract  $5m - 3n$ .

22. From  $9cd - ab$ , take  $2m - 3n - 4y$ .

23. From  $13m - 15$ , take  $-5m + 8$ .

24. From  $7x^2 - 5x + 15$ , take  $-5x^2 + 8x + 15$ .

25. From  $19ab - 2c - 7d$ , take  $3ab - 15c - 8d$ .

26. From  $a$ , take  $b - c$ , and prove the work.

27. From  $11(a + b)$ , take  $5(a + b)$ .

28. From  $17(a - b + x)$ , take  $8(a - b + x)$ .

29. Subtract  $-18(a + b)$  from  $-13(a + b)$ .

30. Subtract  $21(x^2 - y)$  from  $14(x^2 - y)$ .

31. A and B formed an equal partnership and made \$1,000. B's share by right was \$1,000 - \$500; but wishing to withdraw, he agreed to subtract \$100 from his share. What would A's share be?

32. What is the difference of longitude between two places, one of which is 23 degrees due east from the meridian of Washington, the other 37 degrees due west?

REMARK.—The *subtrahend*, in Algebra, is often *greater* than the *minuend*, and the *difference* between a *positive* and *negative* quantity *greater* than either of them. It is thence called *Algebraic Difference*.

**78.** The *Difference* of unlike quantities which have a common letter or letters may be indicated by *enclosing all the other letters*, with their coefficients and signs, in a parenthesis, and *annexing*, or *prefixing* the common letter or letters to the result.

**33.** From  $3am$ , take  $2bm$ .

ANALYSIS.  $3am = 3a$  times  $m$ , and  $2bm = 2b$  times  $m$ ; therefore,  $3am - 2bm = (3a - 2b)m$ , or  $m(3a - 2b)$ . *Ans.*

**34.** From  $2bx^2$ , take  $cx^2 - dx^2$ .

**35.** From  $aby$ , take  $cy + dy - xy$ .

**36.** From  $7a^2$ , take  $ba^2 - ca^2$ .

**37.** From  $abx$ , take  $3cx + dx + mx$ .

**38.** From  $8xy$ , take  $abxy - cxy + dxy$ .

**39.** From  $5ac + bmc$ , take  $3ac - dc$ .

## APPLICATIONS OF THE PARENTHESIS.

**79.** A *parenthesis*, we have seen, shows that the quantities inclosed by it are taken *collectively*, and subjected to the *operation* indicated by the *sign* which precedes it. (Art. 15.)

**80.** A parenthesis having the sign  $+$  prefixed to it, may be removed from an expression, if the *signs* of the included terms remain *unchanged*.

Thus,  $a - b + (c - d + e) = a - b + c - d + e$ . Hence,

**81.** Any number of terms may be inclosed in a parenthesis and the sign  $+$  placed before it, if the *signs* of the inclosed terms remain *unchanged*.

Thus,  $a + b - c + d = a + (b - c + d)$ , or  $a + b + (-c + d)$ .

NOTE.—This principle affords a convenient method of indicating the *addition* of polynomials. (Art. 67.)

78. How subtract unlike quantities having a common letter or letters? 79. What is the object of a parenthesis? 80. How removed when the sign  $+$  is prefixed to it.

**82.** A parenthesis having the sign  $-$  prefixed to it, may be removed by *changing the signs* of all the inclosed terms from  $+$  to  $-$  and  $-$  to  $+$ .

Thus, removing it from the equal expressions,

$$\left. \begin{array}{l} a - b - (-c + d) \\ a - b - (d - c) \end{array} \right\} = a - b + c - d. \quad \text{Hence,}$$

**83.** Any number of terms may be inclosed by a parenthesis, and the sign  $-$  placed before it, if all the signs of the inclosed terms are changed.

Thus,  $a - b + c - d = a - (b - c + d)$ , or  $a - b - (-c + d)$ , etc.

NOTE.—This principle enables us to express a polynomial in different forms without changing its value.

1. How express  $a - x + c$ , using a parenthesis?

Ans.  $a - x + c = a - (x - c)$ , or  $a - (-c + x)$ .

2. How express  $a - b - x - y + z$ , using a parenthesis?

Ans.  $a - b - (x + y - z)$ , or

$a - b - (y + x - z)$ , or

$a - b - (-z + x + y)$ .

**84.** When two or more parentheses occur in the same expression, they are removed by the same rule, beginning with the *interior* parenthesis.

Thus,  $a - [b - c - (d + x) + e] = a - (b - c - d - x + e) = a - b + c + d + x - e$ .

NOTE.—Quantities may be included in more than one parenthesis, by observing the preceding rules.

Remove the parentheses from the following expressions:

3.  $ab - (bc - d)$ .


Ans.  $ab - bc + d$ .

4.  $b - (c - d + m)$ .

5.  $5x - (-y + ab - 4d)$ .

6.  $2a - [b + c - (x + y) - d]$ .

7.  $a - (b - c) - (a - c) + c - (a - b)$ .

 The principles governing the signs in the use and removal of parentheses should be made familiar by practice.

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82. How when the sign  $-$  is prefixed? 83. How inclose terms in a parenthesis with  $-$  prefixed to it?

new

## CHAPTER IV.

### MULTIPLICATION.

**85. *Multiplication*** is finding the *amount* of a quantity taken or added to itself, a given number of times.

Thus, 3 times 4 are 12, and 4 taken 3 times  $(4+4+4) = 12$ .

The ***Multiplicand*** is the quantity to be multiplied.

The ***Multiplier*** is the quantity by which we multiply.

The ***Product*** is the quantity found by multiplication.

**86.** The ***Factors*** of a quantity are the multiplier and multiplicand which produce it.

### PRINCIPLES.

**87. 1°.** *The multiplier must be considered an abstract quantity.*

**2°.** *The product is of the same nature as the multiplicand; for, repeating a quantity does not alter its nature.*

**3°.** *The product of two or more factors is the same in whatever order they are multiplied.*

### CASE I.

**88.** To Multiply a ***Monomial*** by a ***Monomial***.

1. What is the product of *a* multiplied by *c*?

*Ans.*  $a \times c$ , or  $ac$ .

NOTE.—The product of two or more letters, we have seen, is expressed by writing them one after another, either with or without the sign of multiplication between them. (Art. 10.)

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85. Define multiplication. The multiplicand. Multiplier. Product. 86. Factors. 87. Name Prin. 1. Prin. 2. Prin. 3.

2. If 1 ton of iron costs  $a$  dollars, what will  $x$  tons cost?

ANALYSIS.  $x$  tons will cost  $x$  times as much as 1 ton; and  $x$  times  $a$  dollars are  $ax$  dollars. That is,  $a$  dollars are taken  $x$  times, and are equal to  $a+a+a\dots$ , and so on to  $x$  terms.

3. What is the product of  $4a$  by  $2b$ ?

ANALYSIS.—Since each coefficient and each letter in the multiplier and multiplicand is a factor, it follows that the answer must be the product of the coefficients with all the letters of both factors annexed. Hence, the

OPERATION.

$$\begin{array}{r} 4a \\ 2b \\ \hline \text{Ans. } 8ab \end{array}$$

RULE.—*Multiply the coefficients together, and prefix the product to the product of the literal factors.*

Multiply the following quantities:

- |                       |                        |
|-----------------------|------------------------|
| 4. $4ab$ by $5x$ .    | Ans. $20abx$ .         |
| 5. $6bc$ by $7a$ .    | 9. $7xy$ by $8ab$ .    |
| 6. $7abc$ by $5xy$ .  | 10. $6ac$ by $7dx$ .   |
| 7. $8dm$ by $xy$ .    | 11. $9bd$ by $6cm$ .   |
| 8. $9bcd$ by $7xyz$ . | 12. $7xyz$ by $9adf$ . |

## SIGNS OF THE PRODUCT.

89. The investigation of the laws that govern the signs of the product, requires attention to the following

## PRINCIPLES.

- 1°. *Repeating a quantity does not change its sign.*
- 2°. *The sign of the multiplier shows whether the repetitions of the multiplicand are to be added, or subtracted.*

90. If the *Signs* of the factors are *alike*, the sign of the product will be *positive*; if *unlike*, the sign of the product will be *negative*.

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88. How multiply a monomial by a monomial? 89. Name Principle 1. Prin. 2.  
90. If signs of factors are alike, what is the sign of the product? If unlike?

**91. DEMONSTRATION.**—There are four points to be proved:

*First.* That  $+$  into  $+$  produces  $+$ .

Let  $+a$  be the multiplicand and  $+4$  the multiplier. It is plain that  $+a$  taken  $+4$  times is  $+4a$ . (Prin. 1.) The sign of the multiplier being  $+$ , shows that the product  $+4a$ , is to be added, which is done by setting it down without changing its sign. (Art. 66.)

*Second.* That  $-$  into  $+$  produces  $-$ .

Let  $-a$  be multiplied by  $+4$ . Now  $-a$  taken 4 times is  $-4a$ ; for a negative quantity repeated is still *negative*. (Prin. 1.) But the sign before the multiplier being  $+$ , shows that the negative product  $-4a$ , is to be added. This also is done by setting it down without changing its sign. (Art. 66.)

*Third.* That  $+$  into  $-$  produces  $-$ .

Let  $+a$  be multiplied by  $-4$ . We have seen above that  $+a$  taken 4 times is  $+4a$ . But here the sign of the multiplier being  $-$ , shows that the product  $+4a$ , is to be subtracted. This is done by changing its sign from  $+$  to  $-$ , on setting it down. Thus,  $+a \times -4 = -4a$ . (Art. 77.)

*Fourth.* That  $-$  into  $-$  produces  $+$ .

Let  $-a$  be multiplied by  $-4$ . It has also been shown that  $-a$  taken 4 times is  $-4a$ . But the sign of the multiplier being  $-$ , shows that this negative product  $-4a$ , is to be subtracted. This is also done by changing its sign from  $-$  to  $+$ , when we set it down. Thus,  $-a \times -4 = +4a$ . (Art. 77.) Hence, universally,

**92.** *Factors having like signs produce  $+$ , and unlike signs  $-$ .*

13. Multiply  $+4ab$  by  $-7cd$ .

*Ans.*  $-28abcd$ .

14. Multiply  $-5xy$  by  $+9ab$ .

15. Multiply  $+6ab$  by  $+7dc$ .

16. Multiply  $-8xy$  by  $-19abc$ .

17. Multiply  $+18abc$  by  $-23xy$ .

18. Multiply  $-35xy$  by  $-27bcd$ .

91. Prove the first point from the blackboard. The second. Third. Fourth.  
92. Rule for signs.

**93.** When a letter is multiplied into itself, or taken *twice* as a factor, the product is represented by  $a \times a$ , or  $aa$ ; when taken *three times*, by  $aaa$ , and so on, forming a series of powers. But powers, we have seen, are expressed by writing the letter *once* only, with the index above it, at the right hand. (Art. 31.)

19. What is the product of  $aaa$  into  $aa$ ?

**ANALYSIS.**  $aaa \times aa = aaaaaa$ , or  $a^6$ , *Ans.* Now  $aaa = a^3$ , and  $aa = a^2$ ; but adding the exponents of  $a^3$  and  $a^2$  we have  $a^5$ , the same as before. Hence,

**94.** To multiply powers of the same letter together, add their exponents.

**NOTES.**—1. All powers of 1 are 1.

2. When a letter has no exponent, 1 is always understood.

Multiply the following quantities:

20.  $ab^2c^3$  by  $a^2bc$ .

*Ans.*  $a^3b^3c^4$ .

21.  $3a^2b^3x$  by  $2ab^2y$ .

*Ans.*  $6a^3b^4xy$ .

22.  $3xy^2$  by  $5x^2$ .

23.  $ab^m$  by  $ab^n$ .

23.  $6a^2b$  by  $4ab^2$ .

26.  $3xyz$  by  $2xy$ .

24.  $a^4x^2y$  by  $a^2x^2y$ .

27.  $6a^2b^2c$  by  $3a^3b^2c$ .

28. If  $a = 3$ , what is the difference between  $3a$  and  $a^3$ ?

29. If  $x = 4$ , what is the difference between  $4x$  and  $x^4$ ?

**95.** The preceding principles illustrating monomials may be summed up in the following

**RULE.**—Multiply the coefficients and letters of both factors together; to the product prefix the proper sign, and give to each letter its proper index.

**NOTE.**—It is immaterial in what order the factors are taken, but it is more convenient, and therefore customary, to arrange the letters in alphabetical order. (Art. 87, Prin. 3.)

30. Multiply  $-3xy$  by  $-2x$ .

31. Multiply  $6a^2b^2$  by  $-3a^2bc$ .

---

94. How multiply powers of the same letter together? 95. What is the rule for multiplying monomials?

	(32.)	(33.)	(34.)	(35.)
Multiply	$4xy$	$7ab^2$	$8c^2x^3$	$4a^3b^3$
By	$\underline{x^2y}$	$\underline{3a^2b}$	$\underline{5c^2y}$	$\underline{-7ab}$
	(36.)	(37.)	(38.)	(39.)
Multiply	$3xy$	$7abc^2$	$4a^2c^3$	$xyz^2$
By	$\underline{-xy^2}$	$\underline{3abc^2}$	$\underline{-7ac}$	$\underline{xyz}$

### CASE II.

#### 96. To Multiply a *Polynomial* by a *Monomial*.

1. What is the product of  $a + b$  multiplied by  $b$ ?

ANALYSIS.—Multiplying each term of the multiplicand by the multiplier, we have  $a \times b = ab$ , and  $b \times b = b^2$ . The result,  $ab + b^2$ , is the product required. Hence, the

OPERATION.

$$a + b$$

$$\underline{b}$$

$$\text{Ans. } ab + b^2$$

RULE.—Multiply each term of the multiplicand by the multiplier; giving each partial product its proper sign, and each letter its proper index.

Multiply the following quantities:

2.  $bc - ad$  by  $ab$ .

$$\text{Ans. } ab^2c - a^2bd.$$

3.  $3ax^2 + 4cd$  by  $2c$ .

4.  $5ab^2 - 2cd + x$  by  $3ax$ .

5.  $4a^2 - 3ab + m^2$  by  $-2bd$ .

6.  $3a^2 - 4b^2 - 2c^2$  by  $-5a^2c$ .

### CASE III.

#### 97. To Multiply a *Polynomial* by a *Polynomial*.

7. What is the product of  $a + b$  into  $a + b$ ?

ANALYSIS.—Since the multiplicand is to be taken as many times as there are units in the multiplier, the product must be equal to  $a$  times  $a + b$  added to  $b$  times  $a + b$ . Now  $a$  times  $a + b = a^2 + ab$ , and  $b$  times  $a + b = +ab + b^2$ . Hence,  $a + b$  times  $a + b$  must be equal to  $a^2 + 2ab + b^2$ .

OPERATION.

$$a + b$$

$$\underline{a + b}$$

$$a^2 + ab$$

$$+ ab + b^2$$

$$\text{Ans. } a^2 + 2ab + b^2$$

96. How multiply a polynomial by a monomial?

8. Multiply  $b + 2a - 3c$  by  $a + b$ .

**ANALYSIS.**—We multiply each term in the multiplicand by each term in the multiplier, giving to each product the proper sign. (Art. 89.) Finally, we add the partial products, and the result is the answer required.

**OPERATION.**

$$\begin{array}{r}
 2a + b - 3c \\
 \times a + b \\
 \hline
 2a^2 + ab - 3ac \\
 + 2ab \qquad + b^2 - 3bc \\
 \hline
 \text{Ans. } 2a^2 + 3ab - 3ac + b^2 - 3bc
 \end{array}$$

98. The various principles developed in the preceding cases, may be summed up in one

### GENERAL RULE.

*Multiply each term of the multiplicand by each term of the multiplier, giving each product its proper sign, and each letter its proper exponent.*

*The sum of the partial products will be the true product.*

**NOTE.**—For convenience in adding the partial products, like terms should be placed under each other.

Multiply the following quantities:

1.  $2a + b$  by  $3x + y$ .
2.  $3x + 4y$  by  $a - b$ .
3.  $4b - c$  by  $3d - a$ .
4.  $6xy - 2a$  by  $b + c$ .
5.  $3a + 4b - c$  by  $x - y$ .
6.  $5x + 3y + z$  by  $a + b$ .
7.  $7cdx - 3ab$  by  $2m - 3n$ .
8.  $8abc + 4m$  by  $3x - 4y$ .
9. Multiply  $3ab^n$  by  $8a^2b$ . Ans.  $24a^3b^{n+1}$ .
10. Multiply  $-7ax^m$  by  $-8a^2x^n$ . Ans.  $56a^3x^{m+n}$ .
11. Multiply  $3abc^n$  by  $xyz^m$ .
12. Multiply  $acd^m$  by  $11bcd^n$ .
13. Multiply  $-ax^2$  by  $-ax^n$ .
14. Multiply  $x(a + b)^2$  by  $c(a + b)^2$ . (Art. 15.)
15. Multiply  $c(a - b)^3$  by  $5(a - b)^2$ .
16. Multiply  $a(x + y)^m$  by  $bc(x + y)^n$ .
17. Multiply  $3x(a + b)^3$  by  $-(a + b)$ .

---

98. How multiply a polynomial by a polynomial?

**99.** When the polynomials contain *different powers* of the same letter, the terms should be arranged so that the *first* term shall contain the *highest power* of that letter, the second term the *next highest* power, and so on to the last term. This letter is called the *leading letter*.

$$\begin{array}{r}
 (18.) \\
 a^2 + 2ab + b^2 \\
 \hline
 a + b \\
 \hline
 a^3 + 2a^2b + ab^2 \\
 + a^2b + 2ab^2 + b^3 \\
 \hline
 a^3 + 3a^2b + 3ab^2 + b^3, \text{ Ans.}
 \end{array}$$

$$\begin{array}{r}
 (19.) \\
 3a^2b + a^2b \\
 \hline
 4a^2b - 3ab \\
 \hline
 12a^5b^2 + 4a^4b^2 \\
 - 9a^4b^2 - 3a^3b^2 \\
 \hline
 12a^5b^2 - 5a^4b^2 - 3a^3b^2, \text{ Ans.}
 \end{array}$$

20. Multiply  $a^2 - ab + b^2$  by  $a + b$ .
21. Multiply  $a^2 - ab + b^2$  by  $a^2 + ab + b^2$ .
22. Multiply  $x^2 + x + 1$  by  $x^2 - x + 1$ .
23. Multiply  $3x^2 - 2xy + 5$  by  $x^2 + 2xy - 6$ .
24. Multiply  $4ax - 2ay$  by  $6ax + 3ay$ .
25. Multiply  $d + bx$  by  $d + cx$ .

**100.** The *Multiplication* of polynomials may be *indicated* by inclosing each factor in a parenthesis, and writing one after the other.

Thus,  $(a+b)(a+b)$  is equivalent to  $(a+b) \times (a+b)$ .

NOTE.—*Algebraic Expressions* are said to be *developed* or *expanded*, when the operations indicated by their *signs* and *exponents* are performed.

26. Develop the expression  $(a + b)(c + d)$ .  
*Ans.*  $ac + bc + ad + bd$ .
27. Develop  $(x + y)(x - y)$ .
28. Develop  $(a^2 + 1)(a + 1)$ .
29. Expand  $(x^2 + 2xy + y^2)(x + y)$ .
30. Expand  $(a^m + b^n)(a^m + b^n)$ .
31. Expand  $(x + y + z)(x + y + z)$ .

---

99. How arrange different powers of the same letter? 100. How indicate the multiplication of polynomials?

## THEOREMS AND FORMULAS.

**101. THEOREM 1.**—*The Square of the Sum of two quantities is equal to the square of the first, plus twice their product, plus the square of the second.*

1. Let it be required to multiply  $a + b$  into itself.

ANALYSIS.—Each term of the multiplicand being multiplied by each term of the multiplier, we have  $a$  times  $a + b$  and  $b$  times  $a + b$ , the sum of which is  $a^2 + 2ab + b^2$ . Hence, the

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ + ab + b^2 \\ \hline \text{Ans. } a^2 + 2ab + b^2 \end{array}$$

FORMULA.  $(a + b)^2 = a^2 + 2ab + b^2$ .

**102. THEOREM 2.**—*The Square of the Difference of two quantities is equal to the square of the first, minus twice their product, plus the square of the second.*

2. Let it be required to multiply  $a - b$  by  $a - b$ .

ANALYSIS.—Reasoning as before, the result is the same, except the sign of the middle term  $2ab$ , which has the sign  $-$  instead of  $+$ . Hence, the

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ - ab + b^2 \\ \hline \text{Ans. } a^2 - 2ab + b^2 \end{array}$$

FORMULA.  $(a - b)^2 = a^2 - 2ab + b^2$ .

**103. THEOREM 3.**—*The Product of the Sum and Difference of two quantities is equal to the difference of their squares.*

3. Let it be required to multiply  $a + b$  by  $a - b$ .

ANALYSIS.—This operation is similar to the last two; but the terms  $+ab$  and  $-ab$ , in the partial products, being equal, balance each other. Hence, the

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ - ab - b^2 \\ \hline \text{Ans. } a^2 - b^2 \end{array}$$

FORMULA.  $(a + b)(a - b) = a^2 - b^2$ .

**104.** *The product of the sum of two quantities into a third, is equal to the sum of their products.*

4. Let  $x$  and  $y$  be two quantities, whose sum is to be multiplied by  $a$ . Thus,

The product of the sum  $(x+y) \times a = ax+ay$

The sum of the products of  $x \times a + y \times a = ax+ay$

And  $ax+ay = ax+ay$ . Hence, the

**FORMULA.**  $a(x+y) = ax+ay$ .

**105.** *The product of the difference of two quantities into a third, is equal to the difference of their products.*

5. Let  $x$  and  $y$  be two quantities, whose difference is to be multiplied by  $a$ . Thus,

The product of the difference  $(x-y) \times a = ax-ay$

The difference of the products of  $x \times a - y \times a = ax-ay$

And  $ax-ay = ax-ay$ . Hence, the

**FORMULA.**  $a(x-y) = ax-ay$ .

**REMARK.**—The application of the preceding principles is so frequent in algebraic processes, that it is important for the learner to make them very familiar.

Develop the following expressions by the preceding formulas:

- |                         |                          |
|-------------------------|--------------------------|
| 1. $(a+1)(a+1)$ .       | 11. $(4x-1)(4x-1)$ .     |
| 2. $(2a+1)(2a+1)$ .     | 12. $(5b+1)(5b+1)$ .     |
| 3. $(2a-b)(2a-b)$ .     | 13. $(1-x)(1-x)$ .       |
| 4. $(x+y)(x+y)$ .       | 14. $(1+2x)(1+2x)$ .     |
| 5. $(x-y)(x-y)$ .       | 15. $(8b-3a)(8b-3a)$ .   |
| 6. $(1+x)(1-x)$ .       | 16. $(ab+cd)(ab+cd)$ .   |
| 7. $(7y^2-y)(7y^2-y)$ . | 17. $(3a-2y)(3a+2y)$ .   |
| 8. $(4m-3n)(4m+3n)$ .   | 18. $(x^2+y)(x^2-y)$ .   |
| 9. $(x^2-y)(x^2+y)$ .   | 19. $(x-y^2)(x-y^2)$ .   |
| 10. $(1-7x)(1+7x)$ .    | 20. $(2a^2+x)(2a^2-x)$ . |

104. What is the product of the *sum* of two quantities into a third equal to?  
 105. Of the *difference*?

## PROBLEMS.

**106.** Problems requiring each side of the equation to be multiplied by equal quantities.

1. George has 1 third as many pears as apples, and the number of both is 24. How many has he of each?

ANALYSIS.—If  $x$  represents the number of apples, then  $\frac{x}{3}$  will represent the number of pears, and  $x + \frac{x}{3}$  will equal 24, the number of both. The denominator of  $x$  is removed by multiplying each term on both sides of the equation by 3. (Ax. 6.) The result is  $3x + x$ , or  $4x = 72$ . Hence,  $x = 18$ , the apples, and  $18 \div 3 = 6$ , the pears. Hence,

Let  $x =$  No. apples ;

$$\frac{x}{3} = \text{pears.}$$

$$x + \frac{x}{3} = 24$$

$$3x + x = 72$$

$$4x = 72$$

$$\therefore x = 18 \text{ apples.}$$

$$\frac{x}{3} = 6 \text{ pears.}$$

**107.** When a *term* on either side of the equation has a *denominator*, that denominator is removed *by multiplying every term on both sides of the equation by it*. (Ax. 4.)

2. What number is that, 1 seventh of which is 9?

Ans. 63.

3. What number is that, 2 thirds of which are 24?

4. A man being asked how many chickens he had, answered, 3 fourths of them equal 18. How many had he?

5. What number is that, 1 third and 1 fourth of which are 21?

ANALYSIS.—If  $x$  represent the number, then will  $\frac{x}{3} + \frac{x}{4} = 21$ , by the conditions. Multiplying each term on both sides by the denominators 3 and 4 separately, we have  $4x + 3x = 252$ . (Ax. 4.) Uniting the terms,  $7x = 252$ , and  $x = 36$ , Ans.

Let  $x =$  No.

$$\frac{x}{3} + \frac{x}{4} = 21$$

$$4x + 3x = 252$$

$$\therefore x = 36$$

PROOF.  $\frac{1}{3}$  of 36 = 12, and  $\frac{1}{4}$  of 36 = 9. Now,  $12 + 9 = 21$ .

---

107. When a quantity on either side of an equation has a denominator, how remove it?

6. What number is that,  $\frac{2}{3}$  of which exceed  $\frac{1}{2}$  of it by 8?

7. A general lost 840 men in battle, which equaled  $\frac{3}{7}$  of his army. Of how many men did his army consist?

8. If  $\frac{3}{8}$  of a yacht are worth \$360, what is the whole worth?

9. If  $\frac{4x}{5}$  equals 20, to what is  $x$  equal?

10. If  $\frac{5x}{4}$  is equal to 20, to what is  $x$  equal?

11. If  $\frac{3x}{7}$  is equal to 24, to what is  $x$  equal?

12. If  $\frac{4x}{11}$  is equal to 28, to what is  $x$  equal?

13. Henry has 30 peaches, which are  $\frac{5}{6}$  the number of his apples. How many apples has he?

14. A farmer had  $\frac{3}{7}$  as many cows as sheep, and his number of cows was 30. How many sheep had he? How many of both?

✓ 15. Divide 28 pounds into two parts, such that one may be  $\frac{3}{4}$  of the other.

✓ 16. A lad having given  $\frac{1}{3}$  of his plums to one school-mate, and  $\frac{1}{4}$  to another, had 10 left. How many had he at first?

17. What number is that,  $\frac{1}{3}$  and  $\frac{1}{6}$  of which are 21?

18. What number is that,  $\frac{1}{4}$  of which exceeds  $\frac{1}{6}$  by 12?

✓ 19. Divide 36 into two parts, such that one may be  $\frac{2}{3}$  of the other?

✓ 20. One of my apple trees bore  $\frac{3}{7}$  as many apples as the other, and both yielded 21 bushels. How many bushels did each yield?

# CHAPTER V.

## DIVISION.

**108. Division** is finding how many times one quantity is contained in another.

The **Dividend** is the quantity to be divided.

The **Divisor** is the quantity by which we divide.

The **Quotient** is the quantity found by division.

The **Remainder** is a part of the dividend left after division.

**109.** Division is the *reverse* of multiplication, the *dividend* answering to the *product*, the divisor to *one* of the factors, and the *quotient* to the *other*.

## PRINCIPLES.

**110.** 1°. When the divisor is a quantity of the same kind as the dividend, the quotient is an abstract number.

2°. When the divisor is a number, the quotient is a quantity of the same kind as the dividend.

3°. The product of the divisor and quotient is equal to the dividend.

4°. Cancelling a factor of a quantity, divides the quantity by that factor.

## CASE I.

**111. To Divide a Monomial by a Monomial.**

1. What is the quotient of  $abcd$  divided by  $cd$ ?

ANALYSIS.—The divisor  $cd$  is a factor of the dividend; therefore, if we cancel this factor, the other factor  $ab$ , will be the quotient. (Prin. 4.)

OPERATION.

$cd \overline{) abcd}$

Ans.  $ab$ .

108. Define division. The dividend. Divisor. Quotient. Remainder. 109. On what is division the reverse? Explain. 110. Name the first principle. The second. Third. Fourth.

2. What is the quotient of  $18ab$  divided by  $6a$ ?

ANALYSIS.—Dividing the coefficient of the dividend by that of the divisor, and cancelling the common factor  $a$ , we have  $18ab \div 6a = 3b$ , the quotient required. (Prin. 1.) Hence, the

OPERATION.

$$6a \overline{) 18ab}$$

$$\text{Ans. } 3b.$$

RULE.—Divide one coefficient by the other, and to the result annex the quotient of the literal parts.

Divide the following quantities:

$$(3.) \quad 2c \overline{) 4abc}$$

$$(4.) \quad 4b \overline{) 2obxy}$$

$$(5.) \quad 8xy \overline{) 4oxy}$$

$$(6.) \quad 16b \overline{) 32ab}$$

$$(7.) \quad 9m \overline{) 45abm}$$

$$(8.) \quad 2omn \overline{) 6obcmn}$$

$$(9.) \quad 24xy \overline{) 96mnxy}$$

## SIGNS OF THE QUOTIENT.

**112.** The rule for the signs in division is the same as that in multiplication. That is,

If the divisor and dividend have *like signs*, the sign of the quotient will be  $+$ ; if *unlike*, the sign of the quotient will be  $-$ .

Thus,  $+a \times +b = +ab$ ; hence,  $+ab \div +b = +a$ .  
 $-a \times +b = -ab$ ; hence,  $-ab \div +b = -a$ .  
 $+a \times -b = -ab$ ; hence,  $-ab \div -b = +a$ .  
 $-a \times -b = +ab$ ; hence,  $+ab \div -b = -a$ .

Divide the following quantities:

$$10. \quad -32abc \text{ by } -4ab. \quad \text{Ans. } 8c.$$

$$11. \quad 18abx \text{ by } -3z. \quad \text{Ans. } -\frac{6abx}{z}.$$

$$12. \quad 21abc \text{ by } -7ab. \quad 15. \quad 48abc \text{ by } -8ac.$$

$$13. \quad -28bcd \text{ by } -4cd. \quad 16. \quad 63bdfx \text{ by } 9bx.$$

$$14. \quad 35cdm \text{ by } 7cm. \quad 17. \quad -72acgm \text{ by } 8cm.$$

**111.** How divide a monomial by a monomial? **112.** What is the rule for the signs?

• **113.** To Divide *Powers* of the same letter.

18. Let it be required to divide  $a^5$  by  $a^3$ .

**ANALYSIS.**—The term  $a^5 = aaaaa$ , and  $a^3 = aaa$ . Rejecting the factors  $aaa$  from the dividend, the result  $aa$ , or  $a^2$ , is the quotient. Subtracting 3, the index of the divisor, from 5, the index of the dividend, leaves 2, the index of the quotient. That is,  $a^5 \div a^3 = a^2$ . (Arts. 31, 110. Prin. 4.) Hence, the

**RULE.**—*Subtract the index of the divisor from that of the dividend.*

Divide the following quantities:

- |                         |                             |
|-------------------------|-----------------------------|
| 19. $d^7$ by $d^3$ .    | 22. $xyz^{13}$ by $xyz^7$ . |
| 20. $x^{11}$ by $x^5$ . | 23. $16ab^2$ by $4ab$ .     |
| 21. $ac^6$ by $ac^4$ .  | 24. $6xy$ by $3y^2$ .       |

**114.** The preceding principles may be summed up in the following

**RULE.**—*Divide the coefficient of the dividend by that of the divisor; to the result annex the quotient of the literal factors, prefixing the proper sign and giving each letter its proper exponent.*

**PROOF.**—*Multiply the divisor and quotient together, as in arithmetic.*

**NOTE.**—If the letters of the divisor are not in the dividend, the division is *expressed* by writing the divisor *under* the dividend, in the form of a fraction.

25. What is the quotient of  $5x$  divided by  $3y$ ? Ans.  $\frac{5x}{3y}$ .

- |                                    |                                    |
|------------------------------------|------------------------------------|
| 26. $-24a^2b^2c^3 \div -3ab.$      | 32. $32x^4y^3z^2 \div 4x^2yz.$     |
| 27. $-36x^2yz^2 \div 6xyz.$        | 33. $96a^3b^2c \div 12ab.$         |
| 28. $5a^2b^3 \div ab.$             | 34. $84a^5x^3y^2 \div 7a^2xy.$     |
| 29. $-7x^3y^2 \div -xy.$           | 35. $108abx^4 \div 9a^2bx^3.$      |
| 30. $a^4b^3c^2 \div a^3b^2c.$      | 36. $132x^4yz^5 \div 11x^2yz^3.$   |
| 31. $16a^3b^5c^4 \div 8a^2b^4c^3.$ | 37. $121m^4n^2x^3 \div 11m^2nx^3.$ |

113. How divide powers of the same letter? 114. Rule for division of monomials? Proof? If the letters of the divisor are not in the dividend, what is done?

CASE II.

**115. To Divide a *Polynomial* by a *Monomial*.**

1. Divide  $ab + ac + ad$  by  $a$ .

ANALYSIS.—Since the factor  $a$  enters into each term of the dividend, it is plain that each term of the dividend must be divisible by this factor. Hence, the

$$\begin{array}{r} \text{OPERATION.} \\ a \overline{) ab + ac + ad} \\ \text{Ans. } b + c + d \end{array}$$

**RULE.**—*Divide each term of the dividend by the divisor, and connect the results by their proper signs.*

**NOTE.**—If a polynomial which contains the same factor in every term, be divided by the other quantities connected by their signs, the quotient will be *that factor*.

Divide the following quantities:

2.  $6a^3 + 10a^2 - 14a$  by  $2a$ .      *Ans.*  $3a^2 + 5a - 7$ .
3.  $4a^4 - 8a^3 + 12a^2$  by  $-2a^2$ .      *Ans.*  $-2a^2 + 4a - 6$ .
4.  $ab^2 + ac^2 + ad^2$  by  $a$ .
5.  $15x^2y + 25xy$  by  $5xy$ .
6.  $6abc - 2a + 8ab$  by  $2a$ .
7.  $-16by^3 + 4y^3$  by  $-8y$ .
8.  $14x^2y - 7xy^2$  by  $-7xy$ .
9.  $xy^2 + xz - x$  by  $x$ .
10.  $35a + 28b - 42$  by  $-7$ .
11.  $15a^2b - 15a^2$  by  $5a$ .
12.  $16x^3ac + 12acd^2 - 4xa^2c$  by  $-4ac$ .
13.  $4a^4 - 20a^2 + 8ab$  by  $4a$ .
14.  $3ab + 15a^2b - 27a^2bd$  by  $3ab$ .
15.  $8a^2bc - 16ab^2c - 20abc^2$  by  $4abc$ .
16.  $6x(a+b)^2 + 9x^2(a+b)^2$  by  $3x$ .
17.  $15(x-y) + 30(x-y)$  by  $5$ .
18.  $ax^2(b-c) - a^2x(b-c)$  by  $ax$ .
19.  $18a^4(a+b)^2 - 12a^3(a+b)^2$  by  $6a^2(a+b)^2$ .
20.  $a^{n+1} - a^{n+2} + a^{n+3}$  by  $a^n$ .

---

125. How divide a polynomial by a monomial?

## CASE III.

116. To Divide a *Polynomial* by a *Polynomial*.1. Divide  $a^3 + 3a^2b + 3ab^2 + b^3$  by  $a^2 + 2ab + b^2$ .

**ANALYSIS.** — For convenience, we arrange the terms so that the first or leading letter of the divisor shall be the first letter of the dividend. The powers of this letter should be arranged in

OPERATION.	
$a^3 + 3a^2b + 3ab^2 + b^3$	$a^2 + 2ab + b^2$
$a^3 + 2a^2b + ab^2$	$a + b$ Quot.
$a^2b + 2ab^2 + b^3$	
$a^2b + 2ab^2 + b^3$	

order, both in the divisor and dividend, the *highest* power standing *first*, the *next highest next*, and so on. The divisor may be placed on the left of the dividend, or on the right, and the quotient under it, at pleasure.

Proceeding as in arithmetic, we find the first term of the divisor is contained in the first term of the dividend  $a$  times. Placing the  $a$  in the quotient under the divisor, we multiply the whole divisor by it, subtract the product, and to the remainder bring down as many other terms as necessary to continue the operation. Dividing as before,  $a^2$  is contained in  $a^2b$ ,  $+b$  times. Multiplying the divisor by  $+b$  and subtracting the product, the dividend is exhausted; therefore  $a + b$  is the quotient. Hence, the

**RULE.**—I. *Arrange the divisor and dividend according to the powers of one of their letters; and finding how many times the first term of the divisor is contained in the first term of the dividend, place the result in the quotient.*

II. *Multiply the whole divisor by the term placed in the quotient; subtract the product from the dividend, and to the remainder bring down as many terms of the dividend as the case may require.*

*Repeat the operation till all the terms of the dividend are divided.*

**NOTE.**—If there is a remainder after all the terms of the dividend are brought down, *place it over the divisor, and annex it to the quotient*.

---

116. How divide a polynomial by a polynomial? If there is a remainder, what is done with it?

2. Divide  $4a^2 + 4ab + b^2$  by  $2a + b$ . *Ans.*  $2a + b$ .
3. Divide  $x^2 + 2xy + y^2$  by  $x + y$ .
4. Divide  $a^2 - 2ab + b^2$  by  $a - b$ .
5. Divide  $a^3 - 3a^2b + 3ab^2 - b^3$  by  $a - b$ .
6. Divide  $ac + bc + ad + bd$  by  $a + b$ .
7. Divide  $ax + bx - ad - bd$  by  $a + b$ .
8. Divide  $2x^2 + 7xy + 6y^2$  by  $x + 2y$ .
9. Divide  $a^2 - b^2$  by  $a + b$ .
10. Divide  $x^2 - y^2$  by  $x - y$ .
11. Divide  $a^3 - b^3$  by  $a - b$ .
12. Divide  $6a^2 + 13ab + 6b^2$  by  $2a + 3b$ .
13. Divide  $a^2 - a - 6$  by  $a - 3$ .
14. Divide  $a^3 - 3a^2x + 3ax^2 - x^3$  by  $a - x$ .
15. Divide  $6x^2 - 96$  by  $3x - 6$ .
16. Divide  $x^2 + 7x + 10$  by  $x + 2$ .
17. Divide  $x^2 - 5x + 6$  by  $x - 3$ .
18. Divide  $c^2 - 2cx + x^2$  by  $c - x$ .
19. Divide  $a^2 + 2ab + b^2$  by  $a + b$ .
20. Divide  $22(a - b)^2$  by  $11(a - b)$ .

PROBLEMS.

1. A father being asked the age of his son, replied, My age is 5 times that of my son, lacking 4 years; and the sum of our ages is 56 years. How old was each?
2. John and Frank have 60 marbles, the former having 3 times as many as the latter. How many has each?
3. The sum of two numbers is 72, one of which is 5 times the other. What are the numbers?
4. A man divided 57 pears between two girls, giving one 4 times as many as the other, lacking 3. How many did each have?
5. Three boys counting their money, found they had 190 cents; the second had twice as many cents as the first, and the third as many as both the others, plus 4 cents. How many cents had each?

6. A farmer has 9 times as many sheep as cows, and the number of both is 200. How many of each?

7. Divide 57 into two such parts that the greater shall be 3 times the less, plus 3. What are the numbers?

8. Given  $2x + 4x + x - 3 = 60$ , to find  $x$ .

9. A and B are 35 miles apart, and travel toward each other, A at the rate of 4 miles an hour, and B, 3 miles. In how many hours will they meet?

10. Given  $a + 5a + 6a + 2a + 7 = 119$ , to find  $a$ .

11. Given  $8b + 5b + 7b - 10 = 130$ , to find  $b$ .

12. A lad having 60 cents, bought an equal number of pears, oranges, and bananas; the pears being 3 cents apiece, the oranges 4 cents, and the bananas 5 cents. How many of each did he buy?

13. A cistern filled with water has two faucets, one of which will empty it in 5 hours, the other in 20 hours. How long will it take both to empty it?

14. Given  $x + \frac{x}{3} = 45$ , to find  $x$ .

15. What number is that, to the half of which if 3 be added, the sum will be 8?

16. Three boys have 42 marbles; B has twice as many as A, and C three times as many as A. How many has each?

17. If A has 2x dollars, and B twice as many as A, and C twice as many as B, how many have all?

18. Divide 40 into 3 parts, so that the second shall be 3 times the first, and the third shall be 4 times the first.

19. A man divided 60 peaches among 3 boys, in such a manner that B had twice as many as A, and C as many as A and B. How many did each receive?

20. Divide 48 into 3 such parts, that the second shall be equal to twice the first, and the third to the sum of the first and second?

21. What number is that, to three-fourths of which if 5 be added, the sum will be 23?

## CHAPTER VI.

### FACTORING.

**117. *Factors*** are quantities which multiplied together produce another quantity. (Art. 86.)

**118. A *Composite Quantity*** is the product of two or more *integral factors*, each of which is *greater* than a unit.

Thus,  $3a$ ,  $5b$ , also  $x^2y^3$ , are composite quantities.

**119. *Factoring*** is resolving a composite quantity into its factors. It is the converse of multiplication.

**120. An *Exact Divisor*** of a quantity is one that will divide it without a remainder. Hence,

NOTE.—The *Factors* of a quantity are always *exact divisors* of it, and *vice versa*.

**121. A *Prime Quantity*** is one which has no *integral divisor*, except *itself* and 1.

Thus, 5 and 7, also  $a$  and  $b$ , are prime quantities. Hence,

NOTE.—The *least divisor* of a composite quantity is a prime factor.

**122. Quantities are *prime to each other*** when they have no common *integral divisor*, except the unit 1.

Thus, 11 and 15, also  $a$  and  $bc$ , are prime to each other.

**123. A *Multiple*** is a quantity which can be divided by another quantity without a remainder. Hence,

A multiple is a *product* of two or more *factors*.

117. What are factors? 118. A composite quantity? 119. What is factoring?  
120. An exact divisor? 121. A prime quantity? 122. When prime to each other?  
123. A multiple?

## PRINCIPLES.

**124.** 1°. *If one quantity is an exact divisor of another, the former is also an exact divisor of any multiple of the latter.*

Thus, 3 is a divisor of 6; it is also a divisor of  $3 \times 6$ , of  $5 \times 6$ , etc.

2°. *If a quantity is an exact divisor of each of two other quantities, it is also an exact divisor of their sum, their difference, or their product.*

Thus, 3 is a divisor of 9 and 15, respectively; it is also a divisor of  $9 + 15$ , or 24; of  $15 - 9$ , or 6; and of  $15 \times 9$ , or 135.

3°. *A composite quantity is divisible by each of its prime factors, by the product of two or more of them, and by no other quantity.*

Thus, the prime factors of 30 are 2, 3, and 5. Now 30 is divisible by 2, by 3, and by  $2 \times 3$ ; by  $2 \times 5$ ; by  $3 \times 5$ ; by  $2 \times 3 \times 5$ , and by no other number.

## CASE I.

**125.** To Find the Prime Factors of Monomials.

1. What are the prime factors of  $12a^2b$ ?

ANALYSIS.—The coefficient  $12 = 2 \times 2 \times 3$ , and  $a^2b = aab$ . Therefore the prime factors of  $12a^2b$  are  $2 \times 2 \times 3aab$ . Hence, the

RULE.—*Find the prime factors of the numeral coefficients, and annex to them the given letters, taking each as many times as there are units in its exponent.*

NOTE.—In monomials, each letter is a factor. Hence, the prime factors of literal monomials are apparent at sight.

Resolve the following quantities into their prime factors:

- |                    |                               |
|--------------------|-------------------------------|
| 2. $15x^2y^3$ .    | Ans. $3 \times 5 x x y y y$ . |
| 3. $18a^2b^2$ .    | 7. $17x^2y^3z$ .              |
| 4. $20bx^3y^2$ .   | 8. $25ab^2cx^3$ .             |
| 5. $35a^3b^2c^2$ . | 9. $77a^2bcd$ .               |
| 6. $21xy^2z^3$ .   | 10. $65m^2n^3x$ .             |

124. Name Principle 1. Principle 2. Principle 3. 125. How find the prime factors of monomials?

CASE II.

126. To Factor a Polynomial.

1. Resolve  $4a^2b + 8ab - 6ac$  into two factors.

ANALYSIS.—By inspection, we perceive the factor  $2a$  is common to each term; dividing by it, the quotient  $2ab + 4b - 3c$  is the other factor. For convenience, we enclose this factor in a parenthesis, and prefix to it the factor  $2a$ , as a coefficient.

OPERATION.

$$\begin{array}{r} 2a \ ) \ 4a^2b + 8ab - 6ac \\ \underline{2ab + 4b - 3c} \end{array}$$

Ans.  $2a(2ab + 4b - 3c)$

PROOF.—The factor  $(2ab + 4b - 3c) \times 2a = 4a^2b + 8ab - 6ac$ . Hence, the

RULE.—*Divide the polynomial by the greatest common monomial factor; the divisor will be one factor, the quotient the other.* (Art. 115.)

NOTE.—Any common factor, or the product of any two or more common factors, may be taken as a divisor; but the result will vary in form according to the factors employed. (Ex. 2.)

2. Resolve  $a^2b + ab^2$  into two factors, one of which shall be a monomial. Ans.  $ab(a + b)$ ,  $a(ab + b^2)$ , or  $b(a^2 + ab)$ .

3. Factor  $a + ab + ac$ .

Ans.  $a(1 + b + c)$ .

4. Factor  $by + bc + 3bx$ .

5. Factor  $2ax + 2ay - 4az$ .

6. Factor  $3bcx - 6bcx - 3abc$ .

7. Factor  $8dmn - 24dm$ .

8. Factor  $35xm + 14ax$ .

9. Factor  $27bdx - 54dmy$ .

10. Factor  $6a^2b + 9a^2c$ .

11. Factor  $21ax^2y + 35axy$ .

12. Factor  $25 + 15x^2 - 20x^2y^2$ .

13. Factor  $x + x^2 + x^3$ .

14. Factor  $3x + 6 - 9y$ .

15. Factor  $19a^5x - 19a^5$ .

## CASE III.

127. To Resolve a *Trinomial* into two equal Binomial Factors.1. Resolve  $x^2 + 2xy + y^2$  into two equal binomial factors.

ANALYSIS.—Since the *square* of a quantity is the product of two equal factors (Art. 30), it follows that the *square root* of a quantity is one of the two equal factors which produce it. (Art. 32.) Therefore the square root of  $x^2$  is  $x$ , that of  $y^2$  is  $y$ . And since the middle term  $2xy$  is twice the product of these two terms,  $x^2 + 2xy + y^2$  must be the square of the binomial  $x + y$ . Consequently,  $x + y$  is one of the two equal binomial factors.

OPERATION.

$$\begin{aligned}\sqrt{x^2} &= x, & \sqrt{y^2} &= y, \\ \therefore x^2 + 2xy + y^2 &= \\ (x + y)(x + y), & \text{Ans.}\end{aligned}$$

2. Resolve  $x^2 - 2xy + y^2$  into two equal binomial factors.

ANALYSIS.—Reasoning as before, the quantity  $x^2 - 2xy + y^2$  is the square of the residual  $x - y$ . Therefore, the two equal factors must be  $x - y$  and  $x - y$ . Hence, the

OPERATION.

$$\begin{aligned}\sqrt{x^2} &= x, & \sqrt{y^2} &= y \\ \therefore x^2 - 2xy + y^2 &= \\ (x - y)(x - y), & \text{Ans.}\end{aligned}$$

RULE.—Find the square root of each of the square terms, and connect these roots by the sign of the middle term.

NOTE.—A *trinomial*, in order to be resolved into equal *binomial* factors, must have two of its terms squares, and the other term *twice the product* of their square roots. (Art. 101.)

Resolve the following into two equal binomials:

- |                           |                                    |
|---------------------------|------------------------------------|
| 3. $a^2 + 2ab + b^2$ .    | 9. $y^2 + 2y + 1$ .                |
| 4. $x^2 - 2xy + y^2$ .    | 10. $1 - 2c^2 + c^4$ .             |
| 5. $m^2 + 4mn + 4n^2$ .   | 11. $x^{2m} + 2x^m y^n + y^{2n}$ . |
| 6. $16a^2 + 8a + 1$ .     | 12. $4a^{2n} - 4a^n + 1$ .         |
| 7. $49 + 70 + 25$ .       | 13. $a^4 + 2a^2 b^2 + b^4$ .       |
| 8. $4a^2 - 12ab + 9b^2$ . | 14. $a^2 x^4 + 2a x^2 y + y^2$ .   |

CASE IV.

**128.** To Factor a Binomial consisting of the *Difference* of two Squares.

1. Resolve  $4a^2 - 9b^2$  into two binomial factors.

ANALYSIS.—Both of these terms are squares; the root of the first is  $2a$ , that of the second is  $3b$ . But the *difference* of the squares of two quantities is equal to the *product* of their sum and difference. (Art. 103.) Now the sum of these two quantities is  $2a + 3b$ , and the difference is  $2a - 3b$ ; therefore,  $4a^2 - 9b^2 = (2a + 3b)(2a - 3b)$ . Hence, the

OPERATION.

$$\sqrt{4a^2} = 2a$$

$$\sqrt{9b^2} = 3b$$

$$\therefore 4a^2 - 9b^2 =$$

$$(2a + 3b)(2a - 3b), \text{ Ans.}$$

**RULE.**—*Find the square root of each term. The sum of these roots will be one factor, and their difference the other.*

**NOTE.**—This rule is one of the numerous applications of the formula contained in Art. 103.

2. Resolve  $a^2 - x^2$  into two binomial factors.
3. Resolve  $9x^2 - 16y^2$  into two binomial factors.
4. Resolve  $y^2 - 4$  into two binomial factors.
5. Resolve  $9 - x^2$  into two binomial factors.
6. Resolve  $a^2 - 1$  into two binomial factors.
7. Resolve  $1 - b^2$  into two binomial factors.
8. Resolve  $25a^2 - 16b^2$  into two binomial factors.
9. Resolve  $4x^2 - y^2$  into two binomial factors.
10. Resolve  $1 - 16a^2$  into two binomial factors.
11. Resolve  $25 - 1$  into two binomial factors.
12. Resolve  $x^4 - y^4$  into two binomial factors.
13. Resolve  $a^2x^2 - b^2y^2$  into two binomial factors.
14. Resolve  $m^4 - n^4$  into two binomial factors.
15. Resolve  $a^{2m} - b^{2n}$  into two binomial factors.

## CASE V.

**129.** Various classes of examples of *higher powers* may be factored by means of the following

## PRINCIPLES.

**1°.** *The difference of any two powers of the same degree is divisible by the difference of their roots.*

Thus,  $(x^2 - y^2) \div (x - y) = x + y.$

$$(x^3 - y^3) \div (x - y) = x^2 + xy + y^2.$$

$$(x^4 - y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3.$$

$$(x^5 - y^5) \div (x - y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4. \quad \text{—}$$

**2°.** *The difference of two even powers of the same degree is divisible by the sum of their roots.*

Thus,  $(x^2 - y^2) \div (x + y) = x - y.$

$$(x^4 - y^4) \div (x + y) = x^2 - x^2y + xy^2 - y^2.$$

$$(x^6 - y^6) \div (x + y) = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5.$$

**3°.** *The sum of two odd powers of the same degree is divisible by the sum of their roots.*

Thus,  $(x^3 + y^3) \div (x + y) = x^2 - xy + y^2.$


$$(x^5 + y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$$

$$(x^7 + y^7) \div (x + y) = x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6, \text{ etc.}$$

NOTE.—The *indices* and *signs* of the quotient follow regular laws:

1st. The *index* of the *first letter* regularly *decreases* by 1, while that of the *following letter* *increases* by 1.

2d. When the *difference* of two powers is divided by the difference of their roots, the *signs* of all the terms in the quotient are *plus*. When their *sum* or *difference* is divided by the *sum* of their roots, the *odd* terms of the quotient are *plus*, and the *even* terms *minus*.

 If the principles and examples of this Case are deemed too difficult for beginners, they may be deferred until the Binomial Theorem is explained. (Arts. 268-270.)

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129. Recite Prin. 1. Prin. 2. Prin. 3. Note. What is the index of the first letter? Of the following letter? What is said of the signs?

**130. To Factor the *Difference of any two Powers of the same Degree.***

1. Resolve  $x^3 - y^3$  into two factors.

SOLUTION.—The binomial  $(x^3 - y^3) \div (x - y) = x^2 + xy + y^2$ .  $\therefore x - y$  and  $x^2 + xy + y^2$  are the factors. (Prin. 1.) Hence, the

RULE.—*Divide the difference of the powers by the difference of the roots; the divisor will be one factor, the quotient the other.*

Resolve the following into two factors:

2.  $x^3 - 1$ .

4.  $x^3 - 1$ .

3.  $x^6 - y^6$ .

5.  $1 - 36y^2$ .

**131. To Factor the *Difference of two even Powers of the same Degree.***

6. Resolve  $a^4 - b^4$  into two factors.

SOLUTION.—By Prin. 2,  $a^4 - b^4$  is divisible by  $a + b$ . Thus,  $(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3$ , the divisor being one factor, the quotient the other. Hence, the

RULE.—*Divide the difference of the given powers by the sum of their roots; the divisor will be one factor, the quotient the other.* (Art. 129, Prin. 2.)

Resolve the following quantities into two factors:

7.  $b^2 - x^2$ .

10.  $x^4 - 1$ .

8.  $d^4 - z^4$ .

11.  $1 - a^6$ .

9.  $a^6 - b^6$ .

12.  $a^8 - 1$ .

**132. To Factor the *Sum of two odd Powers of the same Degree.***

13. Resolve  $a^3 + b^3$  into two factors.

SOLUTION.—Dividing  $a^3 + b^3$  by  $a + b$ , the factors are  $a + b$  and  $a^2 - ab + b^2$ . (Prin. 3.) Hence, the

RULE.—*Divide the sum of the powers by the sum of the roots; the divisor and quotient are the factors.*

---

130. How factor the difference of any two powers of the same degree? 131. How factor the difference of two even powers of the same degree? 132. The sum of two odd powers of the same degree?

Resolve the following quantities into two factors:

14.  $x^5 + y^5$ .

17.  $1 + y^3$ .

15.  $a^3 + 1$ .

18.  $1 + a^5$ .

16.  $a^5 + 1$ .

19.  $1 + b^7$ .

**133.** It will be observed that in the preceding examples of this Case, binomials have been resolved into *two factors*. These factors may or may not be *prime factors*.

Thus, in Ex. 6,  $a^4 - b^4 = (a+b)(a^3 - a^2b + ab^2 - b^3)$ . But the factor  $(a^3 - a^2b + ab^2 - b^3)$  is a composite quantity  $= (a-b)(a^2 + b^2)$ .

**134.** When a binomial is to be resolved into *prime factors*, it should first be resolved into two factors, one of which is *prime*; then the composite factor should be treated in like manner.

20. Let it be required to find the prime factors of  $a^4 - b^4$ .

SOLUTION.—The  $\sqrt{a^4} = a^2$ , and  $\sqrt{b^4} = b^2$ . (Art. 128.)

Now  $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$ . But  $a^2 - b^2 = (a+b)(a-b)$ . (Art. 104.)

Therefore the prime factors of  $a^4 - b^4$  are  $(a^2 + b^2)(a+b)(a-b)$ .

Resolve the following quantities into their prime factors

21.  $a^4 - 1$ . *Ans.*  $(a^2 + 1)(a + 1)(a - 1)$ .

22.  $1 - y^4$ . *Ans.*  $(1 + y^2)(1 + y)(1 - y)$ .

23.  $x^5 - y^5$ .  
*Ans.*  $(x^2 - xy + y^2)(x^2 + xy + y^2)(x + y)(x - y)$ .

24.  $x^4 - 2x^2y^2 + y^4$ .  
*Ans.*  $(x^2 - y^2)^2 = (x + y)(x + y)(x - y)(x - y)$ .

25.  $x^6 - 1$ .  
*Ans.*  $(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$ .

26.  $a^6 + 2a^3b^3 + b^6$ .  
*Ans.*  $(a + b)(a + b)(a^2 - ab + b^2)(a^2 - ab + b^2)$ .

27.  $a^2 + 9a + 18$ . *Ans.*  $(a + 6)(a + 3)$ .

28.  $4a^2 - 12ab + 9b^2$ . *Ans.*  $(2a - 3b)(2a - 3b)$

## CHAPTER VII.

### DIVISORS AND MULTIPLES.

**135.** A *Common Divisor* is one that will divide two or more quantities without a remainder.

**136.** *Commensurable Quantities* are those which have a *common* divisor.

Thus,  $ab^2$  and  $abc$  are commensurable by  $ab$ .

**137.** *Incommensurable Quantities* are those which have no common divisor. (Art. 122.)

Thus,  $ab$  and  $xyz$  are incommensurable.

**138.** To Find a Common Divisor of two or more Quantities.

1. Find a common divisor of  $abx$ ,  $acy$ , and  $adz$ .

**ANALYSIS.**—Resolving the given quantities into factors, we perceive the factor  $a$ , is common to each quantity, and is therefore a common divisor of them. (Art. 119.) Hence, the

**OPERATION.**

$$abx = a \times b \times x$$

$$acy = a \times c \times y$$

$$adz = a \times d \times z$$

*Ans. a.*

**RULE.**—Resolve each of the given quantities into factors, one of which is common to all.

Find a common divisor of the following quantities:

2.  $3abcd$  and  $9abm$ .

*Ans.  $3ab$ .*

3.  $x^2yz$  and  $2abx$ .

6.  $2ax$ ,  $6bx$ ,  $14cx$ .

4.  $a^2b$ ,  $bcd$ ,  $ab^2xy$ .

7.  $35mn$ ,  $7m^2$ ,  $42m^3x$ .

5.  $2abc$ ,  $acz^2$ ,  $a^3cy$ .

8.  $2a^2b$ ,  $12ab^2$ ,  $6a^3b^2$ .

135. What is a common divisor? 136. Commensurable quantities? 137. Incommensurable quantities? 138. How find a common divisor of two or more quantities?

**139.** The *Greatest Common Divisor* of two or more quantities is the greatest quantity that will divide each of them without a remainder.

NOTES.—1. A *common divisor* of two or more quantities is always a *common factor* of those quantities, and the *g. c. d.\** is their *greatest common factor*.

2. A *common divisor* is often called a *common measure*, and the *greatest common divisor*, the *greatest common measure*.

## PRINCIPLES.

**140.** 1°. The *greatest common divisor* of two or more quantities is the *product* of all their *common prime factors*.

2°. A *common divisor* of two quantities is not altered by *multiplying* or *dividing* either of them by any factor not found in the other.

Thus, 3 is a common divisor of 18 and 6; it is also a common divisor of 18, and of  $(6 \times 5)$  or 30.

3°. The *signs* of a *polynomial* may be changed by dividing it by  $-1$ .

Thus,  $(-3a + 4b - 5c) \div -1 = 3a - 4b + 5c$ . (Art. 112.) Hence,

4°. The *signs* of the *divisor*, or of the *dividend*, or of both, may be changed without changing the *common divisor*.

### 141. To Find the Greatest Common Divisor of Monomials by Prime Factors.

1. What is the *g. c. d.* of  $35acx$ ,  $28abc$ , and  $21ay$ ?

ANALYSIS. — Resolving the given quantities into their prime factors, 7 and  $a$  only, are common to each; therefore their product  $7 \times a$ , is the *g. c. d.* required. (Prin. 1.)

OPERATION.

$$35acx = 5 \times 7 \times a \times c \times x$$

$$28abc = 2 \times 2 \times 7 \times a \times \cancel{b} \times c$$

$$21ay = 3 \times 7 \times a \times y$$

$$\therefore 7 \times a = 7a. \text{ Ans.}$$

139. What is the greatest common divisor of two or more quantities? Note 1. What is true of a common divisor of two or more quantities? Of the *g. c. d.*?

140. Name Principle 1. Principle 2. Principle 3.

\* The initials *g. c. d.* are used for the greatest common divisor.

2. Find the *g. c. d.* of  $4a^3b^2c$ ,  $10a^2b^3$ , and  $14abdx$ .

ANALYSIS.—Resolving these quantities into their prime factors, the factor 2 is common to the coefficients; also,  $a$  and  $b$  are common to the literal parts. Now multiplying these common factors together, we have  $2 \times a \times b = 2ab$ , which is the *g. c. d.* required. (Prin. 1.) Hence, the

OPERATION.

$$4a^3b^2c = 2 \times 2 \times aaabbc$$

$$10a^2b^3 = 2 \times 5 \times aabbb$$

$$14abdx = 2 \times 7 \times abdx$$

$$\text{Ans. } 2 \times a \times b = 2ab$$

**RULE.**—Resolve the given quantities into their prime factors; and the product of the factors common to all, will be the greatest common divisor. (Prin. 1.)

NOTE.—In finding the common prime factors of the literal part, give each letter the least exponent it has in either of the quantities

3. Find the *g. c. d.* of  $6a^2c^3$  and  $9abc$ .

4. Of  $16a^2xy$  and  $18acx^2y$ .

5. Of  $12a^3b^2x^5z^3$  and  $16a^5x^2z^2$ .

6. Of  $6ab^2x^4z^5$ ,  $12a^5x^2z^2$ , and  $18a^5x^3z^3$ .

## 142. To Find the Greatest Common Divisor of Quantities by Continued Division.

1. Required the greatest common divisor of  $30x$  and  $42x$ .

ANALYSIS.—If we divide the greater quantity by the less, the quotient is 1, and  $12x$  remainder. Next, dividing the first divisor  $30x$ , by the first remainder  $12x$ , the quotient is 2 and the remainder  $6x$ . Again, dividing the second divisor by the second remainder, the quotient is 2 and no remainder. The last divisor,  $6x$ , is the *g. c. d.*

OPERATION.

$$30x \overline{) 42x} ( 1$$

$$\underline{30x}$$

$$12x \overline{) 30x} ( 2$$

$$\underline{24x}$$

$$6x \overline{) 12x} ( 2$$

$$\underline{12x}$$

DEMONSTRATION.—Two points are required to be proved :

1st. That  $6x$  is a common divisor of the given quantities.

2d. That  $6x$  is their greatest common divisor.

First. We are to prove that  $6x$  is a common divisor of  $30x$  and  $42x$ . By the last division,  $6x$  is contained in  $12x$ , 2 times. Now as  $6x$  is a

141. How find the *g. c. d.* of monomials by prime factors? Note. In finding the prime factors of the literal part, what exponents are given?

5. What is the *g. c. d.* of  $4x^3 - 6x^2 - 4x + 3$  and  $2x^3 + x^2 + x - 1$ ?

## OPERATION.

1st dividend, $4x^3 - 6x^2 - 4x + 3$	$2x^3 + x^2 + x - 1$	1st divisor.
$4x^3 + 2x^2 + 2x - 2$	2	1st quotient.
2d divisor, $-8x^2 - 6x + 5$	$8x^3 + 4x^2 + 4x - 4$	2d dividend
2d quotient, $-x$	$8x^3 + 6x^2 - 5x$	
3d dividend, $-8x^2 - 6x + 5$	$-2x^2 + 9x - 4$	3d divisor.
$-8x^2 + 36x - 16$	4	3d quotient.
$-21) -42x + 21$		
4th divisor, $2x - 1$	$-2x^2 + 9x - 4$	4th dividend.
4th quotient, $-x + 4$	$-2x^2 + x$	
	$8x - 4$	
	$8x - 4$	

$\therefore 2x - 1$  is the *g. c. d.*

ANALYSIS.—Dividing the greater by the less, the first term of the first remainder,  $-8x^2$ , is not contained in  $2x^3$ , the first term of the second dividend. We therefore multiply this dividend by 4, and it becomes  $8x^3 + 4x^2 + 4x - 4$ , and dividing this by the second divisor, the second remainder is  $-2x^2 + 9x - 4$ . Dividing the preceding divisor by this remainder, we see that the third remainder,  $-42x + 21$ , is not contained in the next dividend. Cancelling the factor  $-21$ , the fourth divisor becomes  $2x - 1$ , the greatest common divisor required.

Find the *g. c. d.* of the following quantities:

6.  $x^3 - y^3$  and  $x^2 - 2xy + y^2$ .
7.  $a^3 + b^3$  and  $a^2 + 2ab + b^2$ .
8.  $b^3 - 4$  and  $b^3 + 4b + 4$ .
9.  $x^2 - 9$  and  $x^3 + 6x + 9$ .
10.  $a^3 - 3a + 2$  and  $a^3 - a - 2$ .
11.  $a^3 + 3a^2 + 4a + 12$  and  $a^3 + 4a^2 + 4a + 3$ .
12.  $x^3 + 1$  and  $x^3 + mx^2 + mx + 1$ .
13.  $a^5 - b^5$  and  $a^2 - b^2$ .
14.  $a^3 - 5ab + 4b^2$  and  $a^3 - a^2b + 3ab^2 - 3b^3$ .
15.  $3x^5 - 10x^3 + 15x + 8$  and  $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$ .

M U L T I P L E S.

**144.** A *Multiple* is a quantity which can be divided by another quantity without a remainder. (Art. 123.)

**145.** A *Common Multiple* is a quantity which can be divided by two or more quantities without a remainder.

Thus, 18a is a common multiple of 2, 3, 6, and 9.

**146.** The *Least Common Multiple* of two or more quantities is the *least* quantity that can be divided by each of them without a remainder.

Thus, 21 is the least common multiple of 3 and 7; 30 is the least common multiple of 2, 3, and 5.

P R I N C I P L E S.

**147.** 1°. *A multiple of a quantity must contain all the prime factors of that quantity.*

Thus, 18 is a multiple of 6, and contains the prime factors of 6, which are 2 and 3.

2°. *A common multiple of two or more quantities must contain all the prime factors of each of the given quantities.*

Thus, 42, a common multiple of 14 and 21, contains all the prime factors of those quantities; viz., 2, 3, and 7.

3°. *The least common multiple of two or more quantities is the least quantity which contains all their prime factors, each factor being taken the greatest number of times it occurs in either of the given quantities.*

Thus, 30 is the least common multiple of 6 and 10, and contains all the prime factors of these quantities; viz., 2, 3, and 5.

---

144. What is a multiple? 145. A common multiple? 146. The least common multiple? 147. Name Principle 1. Principle 2. Principle 3.

### 148. To Find the Least Common Multiple of Monomials by Prime Factors.

1. Find the *l. c. m.*\* of  $15a^4x^3$ ,  $5b^2cx$ , and  $9bc^3z$ .

ANALYSIS.—The prime factors of the coefficients are 5, 3, and 3.

The prime factors of the letter  $a$  are  $a, a, a, a$ , which are denoted by  $a^4$ . In like manner, the prime factors of  $x$  are denoted by  $x^3$ , those of

$b$  by  $b^2$ , and those of  $c$  by  $c^3$ ;  $z$  is prime. Taking each of these factors the greatest number of times it occurs in either of the given quantities, the product,  $45a^4b^2c^3x^3z$ , is the *l. c. m.* required. (Art. 147, Prin. 2.) Hence, the

OPERATION.

$$15a^4x^3 = 3 \times 5 \times a^4 \times x^3$$

$$5b^2cx = 5 \times b^2 \times c \times x$$

$$9bc^3z = 3 \times 3 \times b \times c^3 \times z$$

$$\text{Ans. } 45a^4b^2c^3x^3z.$$

**RULE.**—Resolve the quantities into their prime factors; multiply these factors together, taking each the greatest number of times it occurs in either of the given quantities. The product is the *l. c. m.* required.

Or, Find the least common multiple of the coefficients, and annex to it all the letters, giving each letter the exponent of its highest power in either of the quantities.

NOTE.—In finding the *l. c. m.* of algebraic quantities, it is often more expeditious to arrange them in a horizontal line, then divide, etc., as in arithmetic.

Required the *l. c. m.* of the following quantities:

2.  $9a^3$ ,  $12a^2x^3$ , and  $24ax^2y$ .      '      *Ans.*  $72a^3x^3y$ .
3.  $7ab^2c$ ,  $28bc^3$ , and  $56a^4b^2d$ .
4.  $16x^2y^3z$ ,  $20y^4z$ , and  $8xyz^5$ .
5.  $15a^3b^2c$ ,  $9ab^4c^2$ , and  $18a^2bc^5$ .
6.  $28ab^3$ ,  $14a^2b^4$ ,  $35a^3b^2$ , and  $42a^4b$ .
7.  $21x^4y^3z^2$ ,  $35x^3y^2z^5$ , and  $63xy^3z$ .
8.  $7m^2n^2y$ ,  $12m^3ny^2$ , and  $3mn^2y^4$ .

148. How find the *l. c. m.* of monomials by prime factors? What other method?

\* The initials *l. c. m.* are used for the least common multiple.

**149. To Find the Least Common Multiple of Polynomials.**

9. Required the *l. c. m.* of  $a^3 \div b^3$  and  $a^2 - b^2$ .

**ANALYSIS.** — Resolving the quantities into their prime factors, as in the margin,  $(a+b)$  is common to both, and is their *g. c. d.* (Art. 139.) Now multiply-

ing these factors together, taking each the greatest number of times it occurs in either of the given quantities, the product  $a^4 - a^2b + ab^2 - b^4$  is the *l. c. m.* required. (Art. 148.)

**OPERATION.**

$$\begin{aligned} a^2 - b^2 &= (a+b) \times (a-b) \\ a^3 \div b^3 &= (a+b) \times (a^2 - ab + b^2) \\ (a+b) \times (a-b) \times (a^2 - ab + b^2) &= \\ a^4 - a^2b + ab^2 - b^4. & \text{Ans.} \end{aligned}$$

**SECOND METHOD.**

Since the *g. c. d.* contains all the factors common to both quantities (Art. 147, Prin. 2), it follows if one of them is divided by the *g. c. d.* and the quotient multiplied by the other, the product will be the *l. c. m.* Hence, the

**RULE.**—*Resolve the quantities into their prime factors and multiply these factors together, taking each the greatest number of times it occurs in either of the given quantities. Their product is the l. c. m. required.*

Or, *Find the greatest common divisor of the given quantities, and divide one of them by it. The quotient, multiplied by the other, will be their l. c. m.*

10. Find the *l. c. m.* of  $2a - 1$  and  $4a^2 - 1$ .

**SOLUTION.**—The *g. c. d.* is  $2a - 1$ . Now  $(4a^2 - 1) \div (2a - 1) = (2a + 1)$ ; and  $(2a + 1) \times (2a - 1) = 4a^2 - 1$ , *Ans.*

Find the *l. c. m.* of the following quantities:

11.  $x^2 - y^2$  and  $x^2 - 2xy + y^2$ .

*Ans.*  $x^3 - x^2y - xy^2 + y^3$ .

12.  $a^2 - b^2$  and  $a^3 - b^3$ .

13.  $x^2 - 1$  and  $x^3 + 2x + 1$ .

14.  $2a^2 + 3a - 2$  and  $6a^2 - a - 1$ .

15.  $m^2 + m - 2$  and  $m^3 - 1$ .

(See Appendix, p. 284.)

## CHAPTER VIII.

### FRACTIONS.

**150.** A *Fraction* is one or more of the *equal parts* into which a *unit* is divided.

**151.** Fractions are expressed by *two quantities* called the *numerator* and *denominator*, one of which is written below the other, with a short line between them.

**152.** The *Denominator* is the quantity *below* the line, and shows into *how many* equal parts the unit is divided.

**153.** The *Numerator* is the quantity *above* the line, and shows *how many* parts are taken.

Thus, the expression  $\frac{a}{b}$  shows that the quantity is divided into *b* equal parts, and that *a* of those parts are taken.

**154.** The *Unit* or *Base* of a fraction is the *quantity* divided into equal parts.

**155.** The *Terms* of a fraction are the *numerator* and *denominator*.

**156.** An *Integer* is a quantity which consists of one or more entire units *only*; as *a*, *3a*, *5*, *7*.

**157.** A *Mixed Quantity* is one which contains an integer and a fraction.

Thus,  $a + \frac{x}{c}$  is a mixed quantity.

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150. What is a fraction? 151. How expressed? 152. What does the denominator show? 153. The numerator? 154. What is the base of a fraction? 155. The terms of a fraction? 156. An integer? 157. A mixed quantity?

**158.** Fractions arise from *division*, the numerator being the dividend and the denominator the divisor. Hence,

**159.** The *Value* of a fraction is the *quotient* of the numerator divided by the denominator.

Thus, the value of 6 thirds is  $6 \div 3$ , which is 2; of  $\frac{12m}{4}$  is  $3m$ .

## SIGNS OF FRACTIONS.

**160.** *Every Fraction* has the sign + or —, expressed or understood, before the dividing line.

**161.** The *Dividing Line* has the force of a *vinculum* or *parenthesis*, and the *sign* before it shows that the *value of the whole fraction* is to be added or subtracted.

**162.** *Every Numerator and Denominator* is preceded by the sign + or —, expressed or understood. In this case, the sign affects only the *single term* to which it is prefixed.

**163.** If the *Sign before the Dividing Line* is changed from + to —, or from — to +, the *value* of the fraction is changed from + to —, or from — to +.

Thus,  $a + \frac{bx - cx - dx}{x} = a + b - c - d$ ; but  $a - \frac{bx - cx - dx}{x} = a - b + c + d$ .

**164.** If *all the Signs of the Numerator* are changed, the *value* of the fraction is changed in a corresponding manner.

Thus,  $\frac{+ax + bx}{x} = +a + b$ ; but  $\frac{-ax - bx}{x} = -a - b$ .

158. From what do fractions arise? 159. What is the value of a fraction? 160. What is prefixed to the dividing line of a fraction? 161. What is the force of the dividing line? 162. By what is the numerator and denominator preceded? How far does the force of this sign extend? 163. If the sign before the dividing line is changed, what is the effect? 164. If all the signs of the numerator are changed?

**165.** If *all the Signs of the Denominator* are changed, the *value* is also changed in a corresponding manner.

Thus,  $\frac{ax}{+x} = +a$ ; but  $\frac{ax}{-x} = -a$ . Hence,

**166.** If *any two* of these changes are made at the *same* time, they *will balance* each other, and the value of the fraction will not be *altered*.

Thus,  $\frac{ab}{b} = \frac{-ab}{-b} = -\frac{-ab}{b} = -\frac{ab}{-b} = +a$ .

Also,  $-\frac{ab}{b} = \frac{-ab}{b} = \frac{ab}{-b} = -\frac{-ab}{-b} = -a$ .

### PRINCIPLES.

**167.** The principles for the treatment of fractions in Algebra are the same as those in Arithmetic.

1°. *Multiplying the numerator, or* } *Multiplies the*  
*Dividing the denominator,* } *fraction.*

Thus,  $\frac{2 \times 2}{6} = \frac{4}{6} = \frac{2}{3}$ . And  $\frac{2}{6 \div 2} = \frac{2}{3}$ .

2°. *Dividing the numerator, or* } *Divides the fraction.*  
*Multiplying the denominator,* }

Thus,  $\frac{2 \div 2}{6} = \frac{1}{6}$ . And  $\frac{2}{6 \times 2} = \frac{2}{12} = \frac{1}{6}$ .

3°. *Multiplying, or dividing both* } *Does not change its*  
*terms by the same quantity* } *value.*

Thus,  $\frac{2 \times 2}{6 \times 2} = \frac{4}{12} = \frac{2}{6} = \frac{1}{3}$ . And  $\frac{2 \div 2}{6 \div 2} = \frac{1}{3}$ .

4°. *Multiplying and dividing a* } *Does not change its*  
*fraction by the same quantity* } *value.*

Thus,  $\frac{2 \times 2 \div 2}{6 \times 2 \div 2} = \frac{2}{6}$ .

165. If all the signs of the denominator are changed? 166. If both are changed?  
 167. Name Principle 1. Principle 2. Principle 3. Principle 4.

## REDUCTION OF FRACTIONS.

**168. Reduction of Fractions** is changing their terms without altering the *value* of the fractions.

## CASE I.

**169. To Reduce a Fraction to its Lowest Terms.**

DEF.—The *Lowest Terms* of a fraction are the *smallest terms* in which its numerator and denominator can be expressed. (Art. 122.)

1. Reduce  $\frac{5a^2b^2dx}{15abcx}$  to its lowest terms.

ANALYSIS.—By inspection, we perceive the factors 5, *a*, *b*, and *x* are common to both terms. Cancelling these common factors, the fraction

OPERATION.

$$\frac{5a^2b^2dx}{15abcx} = \frac{abd}{3c}$$

becomes  $\frac{abd}{3c}$ . Now since both terms have been divided by the same quantity, the value of the fraction is not changed. (Art. 167, Prin. 3.) And since these terms have no common factor, it follows that  $\frac{abd}{3c}$  are the lowest terms required. (Art. 122.)

NOTE.—It will be observed that the factors 5, *a*, *b*, and *x* are prime; therefore, the product *5abx* is the *g.c.d.* of the numerator and denominator. (Art. 121.) Hence, the

RULE.—Cancel all the factors common to the numerator and denominator

Or, Divide both terms of the fraction by their greatest common divisor. (Art. 167.)

2. Reduce  $\frac{4abcy}{12abc}$  to its lowest terms.

Ans.  $\frac{y}{3}$

3. Reduce  $\frac{18a^2b}{3ac}$  to its lowest terms.

Ans.  $\frac{6ab}{c}$

---

168. What is reduction of fractions? 169. What are the lowest terms of a fraction? How reduce fractions to the lowest terms?

Reduce the following fractions to the lowest terms:

$$4. \frac{3xy}{9x^2y^2}.$$

$$5. \frac{12a^2bc^2}{4abcd}.$$

$$6. \frac{17b^2cxy}{51b^2cxy}.$$

$$7. \frac{84a^3b^2c^2}{108abx^4y^3}.$$

$$8. \frac{a^2 - b^2}{a^2 + 2ab + b^2}.$$

$$9. \frac{x^2 - y^2}{x^2 - 2xy + y^2}.$$

$$10. \frac{3xy^2 - 3x^2y}{2x^2y - 2xyz}.$$

$$11. \frac{a + bc}{(a + bc) \times x}.$$

$$12. \frac{x^2 - y^2}{x^4 - y^4}.$$

$$13. \frac{ax - x^2}{a^2 - x^2}.$$

$$14. \frac{a - 1}{a^2 - 2a + 1}.$$

$$15. \frac{x + y}{x^2 + 2xy + y^2}.$$

## CASE II.

**170. To Reduce a Fraction to a *Whole or Mixed* Quantity.**

1. Reduce  $\frac{2a + 4b + c}{2}$  to a whole or mixed quantity.

**ANALYSIS.**—Since the value of a fraction is the quotient of the numerator divided by the denominator, it follows that per-

forming the division indicated will give the answer required. Now 2 is contained in  $2a$ ,  $a$  times; in  $4b$ ,  $2b$  times. Placing the remainder  $c$  over the denominator, we have  $a + 2b + \frac{c}{2}$ , the mixed quantity required. Hence, the

**OPERATION.**

$$\frac{2a + 4b + c}{2} = a + 2b + \frac{c}{2}.$$

**RULE.**—*Divide the numerator by the denominator, and placing the remainder over the divisor, annex it to the quotient.*

**NOTE.**—This rule is based upon the principle that both terms are *divided* by the *same* quantity. (Art. 167, Prin. 3.)

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170. How reduce a fraction to a whole or mixed quantity? *Note.* Upon what principle is this rule based?

Reduce the following to whole or mixed quantities:

$$2. \frac{ax - x^2}{x}.$$

$$6. \frac{a^2 - 2ab + b^2}{a - b}.$$

$$3. \frac{ab - b^2}{a}.$$

$$7. \frac{b^2 + ab^2}{b^2 - ab}.$$

$$4. \frac{b^2 - c^2}{b + c}.$$

$$8. \frac{a^3 + a^2 - ax^2}{a^2 - ax}.$$

$$5. \frac{b^2 + c^2}{b - c}.$$

$$9. \frac{12x^2 + 4x - 3y}{4x}.$$

### CASE III.

**171. To Reduce a Mixed Quantity to an *Improper* Fraction.**

1. Reduce  $a + \frac{b}{3}$  to the form of a fraction.

ANALYSIS.—Since in 1 unit there are three thirds, in  $a$  units there must be  $a$  times 3 thirds, or  $\frac{3a}{3}$ ; and  $\frac{3a}{3} + \frac{b}{3} = \frac{3a+b}{3}$ , the fraction required. Hence, the

OPERATION.

$$a + \frac{b}{3} = \frac{3a}{3} + \frac{b}{3}$$

$$\frac{3a}{3} + \frac{b}{3} = \frac{3a+b}{3}$$

**RULE.**—Multiply the integer by the denominator; to the product add the numerator, and place the sum over the denominator. (Art. 65.)

NOTES.—1. An integer may be reduced to the form of a fraction by making 1 its denominator. Thus,  $a = \frac{a}{1}$ .

2. If the sign before the dividing line is — and the denominator is removed, all the signs of the *numerator* must be changed. (Arts. 163, 82.)

Reduce the following to improper fractions:

$$2. ab - \frac{cd}{d}.$$

$$\text{Ans. } \frac{abd - cd}{d} = ab - c.$$

$$3. 3x + \frac{xy - b}{y}.$$

$$6. x - 1 + \frac{1 - x}{1 + x}.$$

$$4. 5d + \frac{a - c}{2b}.$$

$$7. 4a - \frac{a - b}{3c}.$$

$$5. a + b + \frac{2x}{a + b}.$$

$$8. 8x + \frac{3a - x^2}{5x}.$$

## CASE IV.

**172. To Reduce an Integer to a Fraction having any required Denominator.**1. Reduce  $3a$  to fifths.

ANALYSIS.—Since in  $1a$  there are 5 fifths, in  $3a$  there must be 3 times 5 fifths, or  $\frac{15a}{5}$ .

OPERATION.

$$3a = \frac{3a}{1}$$

Or, reducing the integer  $3a$  to the form of a fraction, it becomes  $\frac{3a}{1}$ ; multiplying both

$$\frac{3a \times 5}{1 \times 5} = \frac{15a}{5}$$

terms by the required denominator, we have  $\frac{15a}{5}$ . Hence, the

**RULE.**—*Multiply the integer by the required denominator, and place the product over it.*

2. Reduce  $2x$  to a fraction having  $6m$  for its denominator.3. Reduce  $6ax$  to a fraction having  $4ab$  for its denominator.4. Reduce  $3a + 4b$  to a fraction having  $6c^2$  for its denominator.5. Reduce  $x - y$  to a fraction having  $x + y$  for its denominator.6. Reduce  $2x^2y$  to a fraction having  $3a^2 - 2b$  for its denominator.**173. To Reduce a Fraction to any Required Denominator.**1. Change  $\frac{a}{3}$  to a fraction whose denominator is 12.

ANALYSIS.—Dividing 12, the required denominator, by the given denominator 3, the quotient is 4. Multiplying both terms of the given fraction by the quotient 4, the

OPERATION.

$$12 \div 3 = 4$$

$$\text{Ans. } \frac{a \times 4}{3 \times 4} = \frac{4a}{12}$$

result,  $\frac{4a}{12}$ , is the fraction required. Hence, the

**RULE.**—*Divide the required denominator by the denominator of the given fraction, and multiply both terms by the quotient.*

Reduce the following to the required denominators:

2. Reduce  $\frac{3a}{7}$  to thirty-fifths.

SOLUTION.  $35 \div 7 = 5$ . Now  $\frac{3a \times 5}{7 \times 5} = \frac{15a}{35}$ . *Ans.*

3. Reduce  $\frac{b}{c}$  to the denominator  $ac$ .

4. Reduce  $\frac{3a}{7}$  to the denominator  $49a$ .

5. Reduce  $\frac{x+y}{x-y}$  to the denominator  $x^2 - 2xy + y^2$ .

6. Reduce  $\frac{4a}{x+y}$  to the denominator  $8a^2(x+y)^2$ . —

## COMMON DENOMINATORS.

**174.** A *Common Denominator* is one that belongs equally to *two or more* fractions.

## PRINCIPLES.

1°. A common denominator is a multiple of each of the denominators; for every quantity is a divisor of itself and of every multiple of itself. (Art. 124, Prin. 1.) Hence,

2°. The least common denominator is the least common multiple of all the denominators.

## CASE V.

**175.** To Reduce Fractions to Equivalent Fractions having a *Common Denominator*.

1. Reduce  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{x}{y}$ , to equivalent fractions having a common denominator.

173. How reduce a fraction to any required denominator? 174. What is a common denominator? Principle 1? Principle 2?

**SOLUTION.**—Multiplying the denominators  $b$ ,  $d$ , and  $y$  together, we have  $bdy$ , which is a common denominator.

$$b \times d \times y = bdy \quad \text{The common denominator.}$$

$$a \times d \times y = ady$$

$$c \times b \times y = bcy$$

$$x \times b \times d = bdx$$

} The new numerators.

$$\text{Ans.} \quad \frac{a}{b} = \frac{ady}{bdy}, \quad \frac{c}{d} = \frac{bcy}{bdy}, \quad \frac{x}{y} = \frac{bdx}{bdy}.$$

To reduce the given fractions to this denominator, we multiply each numerator by all the denominators except its own, and place the results over the common denominator. Hence, the

**RULE.**—*Multiply all the denominators together for a common denominator, and each numerator into all the denominators except its own, for the new numerators.*

**NOTES.**—1. It is advisable to reduce the fractions to *their lowest terms*, before the rule is applied. (Art. 169.)

2. This rule is based on the principle, that the *value* of a fraction is not changed by multiplying both its terms by the same quantity. (Art. 167, Prin. 3.)

Reduce the following to equivalent fractions having a common denominator:

$$2. \quad \frac{a}{c}, \frac{x}{y}, \frac{3}{4}.$$

$$\text{Ans.} \quad \frac{4ay}{4cy}, \frac{4cx}{4cy}, \frac{3cy}{4cy}.$$

$$3. \quad \frac{c}{d}, \frac{b}{x}, \frac{2d}{4}.$$

$$8. \quad x, \frac{2a}{b}, \frac{c+1}{d}.$$

$$4. \quad \frac{a}{2x}, \frac{b}{c^2}, \frac{x}{y}.$$

$$9. \quad \frac{2}{3}, \frac{a}{b^2}, \frac{c+d}{c-d}.$$

$$5. \quad \frac{2a}{3b}, \frac{x}{a+b}.$$

$$10. \quad \frac{xy}{z}, \frac{1}{2}, \frac{2a}{b}.$$

$$6. \quad \frac{x-y}{x+y}, \frac{x+y}{x-y}.$$

$$11. \quad \frac{2a}{4}, 3, \frac{x^2+y^2}{x+y}.$$

$$7. \quad \frac{a+b}{3}, \frac{5a-1}{a}.$$

$$12. \quad \frac{a-x}{a+x}, \frac{a+x}{a-x}.$$

175. How reduce fractions to equivalent fractions having a common denominator? *Note.* Upon what principle is this rule based?

## CASE VI.

**176. To Reduce Fractions to the *Least Common Denominator*.**

1. Reduce  $\frac{a}{x}$ ,  $\frac{b}{xy}$ , and  $\frac{d}{yz}$  to the *l. c. d.*

SOLUTION.—The *l. c. m.* of the denominators is  $xyz$ . (Art. 148.)

$$\begin{array}{lcl}
 xyz = l. c. d. & & \frac{a \times yz}{x \times yz} = \frac{ayz}{xyz} \\
 \left. \begin{array}{l} xyz \div x = yz \\ xyz \div xy = z \\ xyz \div yz = x \end{array} \right\} \begin{array}{l} \text{The} \\ \text{multipliers.} \end{array} & & \left. \begin{array}{l} \frac{b \times z}{xy \times z} = \frac{bz}{xyz} \\ \frac{d \times x}{yz \times x} = \frac{dx}{xyz} \end{array} \right\} \begin{array}{l} \text{The} \\ \text{fractions} \\ \text{required.} \end{array}
 \end{array}$$

To change the given fractions to others whose denominator is  $xyz$ , we multiply each numerator by the quotient arising from dividing this multiple by its corresponding denominator. Hence, the

**RULE.—I.** Find the least common multiple of all the denominators for the least common denominator.

**II.** Divide this multiple by the denominator of each fraction, and multiply its numerator by the quotient.

**NOTE.**—All the fractions must be reduced to their *lowest* terms before the rule is applied.

Reduce the following fractions to the *l. c. d.* :

- |  |  |
|--|--|
| 2. $\frac{a}{2b}, \frac{bc}{x}, \frac{y}{4c}$ .                  | 8. $\frac{a+b}{a-b}, \frac{a-b}{a+b}, \frac{a^2+b^2}{a^2-b^2}$ . |
| 3. $\frac{cd}{ab}, \frac{2x}{3a}, \frac{xy}{ac}$ .               | 9. $\frac{2(x+y)}{3(x+y)}, \frac{a}{xy}, \frac{ab}{6(x+y)}$ .    |
| 4. $\frac{a}{2}, \frac{b}{3}, \frac{c}{4}, \frac{x}{y}$ .        | 10. $\frac{d}{ab^3}, \frac{x}{a^2b}$ .                           |
| 5. $\frac{a^2c}{ab}, \frac{2cd}{b^2c}, \frac{x^2y}{4bc}$ .       | 11. $\frac{x}{ac}, \frac{m}{b^2c}, \frac{y}{c^2d}$ .             |
| 6. $\frac{2ab}{3ac}, \frac{3}{4}, \frac{x}{a^2c}, \frac{1}{8}$ . | 12. $\frac{x}{y^2}, \frac{a+b}{xy}, \frac{d}{xz}$ .              |
| 7. $\frac{2a}{4b}, \frac{cd}{bc}, \frac{x^2y}{bcx}$ .            | 13. $\frac{m+n}{3a^2}, \frac{m-n}{2ax^2}, \frac{m^2}{4cx}$ .     |

## ADDITION OF FRACTIONS.

**177.** When fractions have a *common denominator*, their *numerators* express *like parts* of the same unit or base, and are *like quantities*. (Art. 43.)

**178.** To Add Fractions which have a *Common Denominator*.

1. What is the sum of  $\frac{3}{8}$ ,  $\frac{7}{8}$ , and  $\frac{5}{8}$ ?

SOLUTION.  $\frac{3}{8}$  and  $\frac{7}{8}$  are  $\frac{10}{8}$ , and  $\frac{5}{8}$  are  $\frac{15}{8}$ , *Ans.* Hence, the

RULE.—Add the numerators, and place the sum over the common denominator.

2. Add  $\frac{7b}{m}$ ,  $\frac{4b}{m}$ , and  $\frac{8b}{m}$ . *Ans.*  $\frac{19b}{m}$ .

3. Add  $\frac{3ac}{2xy}$ ,  $\frac{11ac}{2xy}$ ,  $\frac{8ac}{2xy}$ , and  $\frac{5ac}{2xy}$ .

4. Add  $\frac{7dxz}{5abc}$ ,  $\frac{17dxz}{5abc}$ ,  $\frac{11dxz}{5abc}$ , and  $\frac{4dxz}{5abc}$ .

5. Add  $\frac{2b+c}{x}$  to  $\frac{3b-c}{x}$ . 6. Add  $\frac{3a+4b}{c}$  to  $\frac{4a-2b}{c}$ .

**179.** To Add Fractions which have *Different Denominators*.

7. What is the sum of  $\frac{a}{b}$ ,  $\frac{c}{d}$ , and  $\frac{m}{x}$ ?

ANALYSIS.—Since these fractions have *different denominators*, their numerators cannot be added in their present form. (Art. 66.) We therefore reduce them to a *common denominator*, then add the numerators. (Art. 175.) Hence, the

$$b \times d \times x = bdx, c. d.$$

$$\frac{a}{b} = \frac{adx}{bdx} \quad \frac{c}{d} = \frac{bcx}{bdx}$$

$$\frac{m}{x} = \frac{bdm}{bdx}$$

$$\frac{adx + bcx + bdm}{bdx}, \text{ Ans.}$$

RULE.—Reduce the fractions to a common denominator, and place the sum of the numerators over it.

NOTE.—All answers should be reduced to the lowest terms.

177. When fractions have a common denominator, what is true of the numerators?

178. How add such fractions? 179. How, when they have different denominators?

Find the sum of the following fractions:

- |  |  |
|--|--|
| 8. $\frac{2x}{y} + \frac{2y}{x} + \frac{2xy}{m}$ | Ans. $\frac{2x^2m + 2y^2m + 2x^2y^2}{mxy}$         |
| 9. $\frac{3a}{4} + \frac{x}{5} + \frac{y}{3}$    | 15. $\frac{cd}{3x} + \frac{4y}{2d} + \frac{bx}{5}$ |
| 10. $\frac{2x}{3} + \frac{1}{2a} + \frac{3y}{4}$ | 16. $\frac{a}{d} - \frac{2n + d}{3h}$              |
| 11. $\frac{a}{b+c} + \frac{x}{b-c}$              | 17. $\frac{a}{y} + \frac{d}{-m}$                   |
| 12. $\frac{x+y}{2xy} + \frac{x-y}{xy}$           | 18. $\frac{-x}{y} + \frac{-h}{m-n}$                |
| 13. $\frac{2+x}{y} + \frac{3+ax}{ay}$            | 19. $\frac{-4}{2} + \frac{-16}{7-3}$               |
| 14. $\frac{a}{x+y} + \frac{ab}{x-y}$             | 20. $\frac{4a}{b} + \frac{6c}{d} - \frac{3m}{3x}$  |

### 180. To Add *Mixed* Quantities.

1. What is the sum of  $a + \frac{b}{c}$  and  $d - \frac{m}{n}$ ?

SOLUTION.—Adding the integral and fractional parts separately, the result is  $a + d + \frac{bn - cm}{cn}$ , the sum required. Hence, the

RULE.—*Add the integral and fractional parts separately, and unite the results.* (Art. 179.)

NOTE.—Mixed quantities may be reduced to improper fractions, and then be added by the rule. (Art. 171.)

- What is the sum of  $a + \frac{b}{2}$  and  $c + \frac{d}{x}$ ?
- What is the sum of  $x + \frac{a}{b}$  and  $\frac{-x+d}{m-y}$ ?
- What is the sum of  $3d - \frac{xy+z}{2}$  and  $a + \frac{b-c}{1}$ ?
- What is the sum of  $5x + \frac{a}{b}$  and  $\frac{-y}{2}$ ?

---

180. How add mixed quantities? Note. How else may they be added?

**181.** To Incorporate an *Integer* with a Fraction.

6. Incorporate the integer  $ab$  with the fraction  $\frac{c-d}{3x+y}$ .

SOLUTION.—Reducing  $ab$  to the denominator of the fraction, we have  $ab = \frac{3abx+aby}{3x+y}$ ; and  $\frac{3abx+aby}{3x+y} + \frac{c-d}{3x+y} = \frac{3abx+aby+c-d}{3x+y}$ , *Ans.*  
Hence, the

RULE.—Reduce the integer to the denominator of the fraction, and place the sum of the numerators over the given denominator. (Art. 172.)

7. Incorporate the integer  $3d$  with the fraction  $\frac{2a}{b}$ .

8. Incorporate  $-4y$  with  $\frac{a+b}{c}$ .

9. Incorporate  $-a$  with  $\frac{x+y}{a}$ .

10. Incorporate  $3x+y$  with  $\frac{a-b}{x-y}$ .

11. Incorporate  $-a+5b$  with  $\frac{x-y}{a-b}$ .

12. Incorporate  $2x+2y$  with  $\frac{a+b}{x-1}$ .

## SUBTRACTION OF FRACTIONS.

**182.** The *numerators* of fractions which have a *common denominator*, we have seen, are *like quantities*. (Art. 177.) Hence, they may be *subtracted* one from another as integers.

1. Subtract  $\frac{5}{8}$  from  $\frac{7}{8}$ .

SOLUTION.  $\frac{7}{8} - \frac{5}{8} = \frac{7-5}{8} = \frac{2}{8}$  or  $\frac{1}{4}$ . *Ans.*  $\frac{1}{4}$ .

2. From  $\frac{a}{b}$  subtract  $\frac{c}{b}$ . *Ans.*  $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ .

181. How incorporate an integer with a fraction? 182. What is true of the *numerators* of fractions having a common denominator? How subtract such fractions?

3. From  $\frac{13abc}{d}$  subtract  $\frac{4abc}{d}$ .

4. From  $\frac{17xyz}{a}$  subtract  $\frac{5cd}{a}$ .

**183. To Subtract Fractions which have *Different Denominators*.**

5. From  $\frac{7a}{3}$  subtract  $\frac{3a}{2}$ .

ANALYSIS.—Since these fractions have different denominators, they cannot be subtracted one from the other in their present form. We therefore reduce them to a *common denominator*, which is 6, and place the difference of the numerators over it. Hence, the

$$3 \times 2 = 6, \text{ c. d.}$$

$$\frac{7a}{3} = \frac{14a}{6}$$

$$\frac{3a}{2} = \frac{9a}{6}$$

$$\frac{14a}{6} - \frac{9a}{6} = \frac{5a}{6}, \text{ Ans.}$$

**RULE.**—Reduce the fractions to a common denominator, and subtract the numerator of the subtrahend from that of the minuend, placing the difference over the common denominator.

NOTES.—1. The *integral* and *fractional* parts of mixed quantities should be subtracted separately, and the results be united.

Or, *mixed quantities* may be reduced to improper fractions, and then be subtracted by the rule. (Art. 171.)

2. A fraction may be subtracted from an integer, or an integer from a fraction, by reducing the integer to the given denominator, and then applying the rule.

6. From  $\frac{5ab}{x}$ , take  $\frac{3cd}{y}$ . Ans.  $\frac{5aby - 3cdx}{xy}$

7. From  $\frac{a}{m}$ , take  $\frac{d-b}{y}$ .

8. From  $\frac{b-d}{m}$ , take  $-\frac{b}{y}$ .

9. From  $\frac{a+3d}{4}$ , take  $\frac{3a-2d}{3}$ .

---

183. How when they have different denominators? *Note 1.* How subtract mixed quantities? *Note 2.* How a fraction from an integer, or an integer from a fraction?

10. From  $\frac{h}{m}$ , take  $-\frac{h+d}{y}$ .

11. From  $\frac{h}{y}$ , take  $m$ .

12. From  $4a + \frac{b}{c}$ , take  $3a - \frac{h}{d}$ .

13. From  $a + \frac{b}{2}$ , take  $\frac{d-b}{3}$ .

14. From  $\frac{a}{b-x}$ , take  $\frac{c}{d+y}$ .

15. From  $a' - \frac{x}{y}$ , take  $\frac{3d}{2}$ .

16. From  $\frac{x-y}{10}$ , take  $\frac{a-b}{x+y}$ .

17. From  $x - \frac{y-c}{2}$ , take  $\frac{x-y}{3} - a$ .

## MULTIPLICATION OF FRACTIONS.

### CASE I.

#### 184. To Multiply a Fraction by an *Integer*.

1. Multiply  $\frac{a}{b}$  by  $m$ .

ANALYSIS.—Multiplying the numerator of the fraction by the integer, the product is  $am$ .  
(Art. 167, Prin. 1.)

OPERATION.

$$\frac{a}{b} \times m = \frac{am}{b}.$$

2. What is the product of  $\frac{a}{bx} \times x$ ?

ANALYSIS.—A fraction is multiplied by dividing its denominator; therefore, if we divide  $bx$  by  $x$ , the result will be the product required. (Art. 167, Prin. 1.) Hence, the

$$\frac{a}{bx} \times x = \frac{a}{bx \div x}$$

$$\frac{a}{bx \div x} = \frac{a}{b}.$$

RULE.—*Multiply the numerator by the integer.*

Or, *Divide the denominator by it.*

NOTES.—1. A fraction is multiplied by a *quantity equal* to its denominator, by *cancelling* the denominator. (Art. 110, Prin. 4.)

2. A fraction is also multiplied by any *factor* in its *denominator*, by *cancelling that factor*. (Art. 110, Prin. 4.)

Find the product of the following quantities:

$$3. \quad \frac{3x}{a-b} \times (a-b).$$

$$\text{Ans. } 3x.$$

$$4. \quad \frac{ab}{cd} \times d.$$

$$\text{Ans. } \frac{ab}{c}.$$

$$5. \quad \frac{h+3d}{3+m} \times (3+m).$$

$$12. \quad \frac{a-b}{6} \times (12x+18).$$

$$6. \quad \frac{ab}{24} \times 6.$$

$$13. \quad \frac{abc}{d-x} \times (d-x).$$

$$7. \quad \frac{2x-3y}{15c+4d} \times (3c+2d).$$

$$14. \quad \frac{a+b}{20x} \times 4x.$$

$$8. \quad a \times \frac{bc}{3x} \times 6x.$$

$$15. \quad \frac{2x+y}{40z-10} \times (8z-2).$$

$$9. \quad \frac{a+b}{20x+25xy} \times 5x.$$

$$16. \quad \frac{3c-d}{20} \times 15.$$

$$10. \quad \frac{a+ab}{bc+c} \times 2ac.$$

$$17. \quad \frac{3x}{y-1} \times (y^2-1).$$

$$11. \quad \frac{2x+3}{5} \times 20x.$$

$$18. \quad \frac{m^2}{x^2-x^2} \times (x+z).$$

## CASE II.

### 185. To Multiply a Fraction by a Fraction.

1. What is the product of  $\frac{a}{c}$  by  $\frac{d}{m}$ .

ANALYSIS.—Multiplying the numerator of the fraction  $\frac{a}{c}$  by  $d$ , the numerator of the multiplier, we have  $\frac{ad}{c}$ . But the multiplier is  $\frac{d}{m}$ ; hence the product  $\frac{ad}{c}$  is  $m$  times too large.

To correct this, multiply the denominator by  $m$ . (Art. 167, Prin. 2.)

OPERATION.

$$\frac{a}{c} \times d = \frac{ad}{c}$$

$$\frac{ad}{c} \times m = \frac{ad}{cm}$$

$$\frac{a}{c} \times \frac{d}{m} = \frac{ad}{cm}$$

Note 1. How is a fraction multiplied by a quantity equal to its denominator?  
Note 2. How by any factor in its denominator?

2. Required the product of  $\frac{2ab}{6cd}$  multiplied by  $\frac{cm}{ax}$ .

ANALYSIS.—The factors 2,  $c$ , and  $c$  are common to each term of the given fractions. Cancelling these common factors, the result is  $\frac{bm}{3dx}$ , the product required (Art. 167, Prin. 3.) Hence, the

OPERATION.

$$\frac{2ab}{6cd} \times \frac{cm}{ax} = \frac{2abcm}{6acdx}$$

$$\frac{2abcm}{6acdx} = \frac{bm}{3dx}, \text{ Ans.}$$

RULE.—Cancel the common factors; then multiply the numerators together for the new numerator, and the denominators for the new denominator.

NOTES.—1. *Mixed quantities* should be reduced to *improper fractions*, and then be multiplied as above.

Or, the *fractional* and *integral* parts may be multiplied separately, and the results be united.

2. Cancelling the common factors *shortens* the operation, and gives the answer in the *lowest* terms.

3. The word *of* in compound fractions has the force of the sign  $\times$ . Therefore, reducing compound fractions to simple ones is the same as multiplying the fractions together. Thus,  $\frac{1}{2}$  of  $\frac{2}{3}$  =  $\frac{1}{2} \times \frac{2}{3}$  =  $\frac{1}{3}$ .

Find the products of the following fractions:

3.  $\frac{2x}{y} \times \frac{3xy}{2d} \times \frac{2dy}{x}$ .

6.  $\frac{1}{a+3x} \times \frac{3}{8}$ .

4.  $\frac{bc}{a} \times \frac{x}{by} \times \frac{d}{c}$ .

7.  $\frac{(a+m) \times h}{3x} \times \frac{4y}{(a+m) \times c}$

5.  $\frac{x-y}{yz} \times \frac{x+y}{y+z}$ .

8.  $\frac{a+b}{c^2} \times \frac{cd}{x}$ .

9. What is the product of  $\frac{3x}{x-y}$  by  $\frac{x^2-y^2}{4}$ ?

SOLUTION.—Factor and cancel. Ans.  $\frac{3x}{4}(x+y)$ .

---

185. How multiply a fraction by a fraction? Note 1. How mixed quantities? Note 2. How shorten the operation? Note 3. What is the force of the word *of* in compound fractions?

$$10. \text{ Multiply } \frac{2a}{x-y} \text{ by } \frac{x^2-y^2}{3d}. \quad \text{Ans. } \frac{2a(x+y)}{3d}.$$

$$11. \text{ Multiply } \frac{2x-y}{4x} \text{ by } \frac{6x-2y}{y^2-2xy}.$$

$$12. \text{ Multiply } \frac{4a-2b}{b^2-2ab} \text{ by } \frac{2a-b}{6a}.$$

$$13. \text{ Multiply } a + \frac{a+b}{x} \text{ by } \frac{ax}{by}.$$

$$14. \text{ Multiply } x + \frac{2x^2}{xy} \text{ by } \frac{x+y}{x^2}.$$

$$15. \text{ Multiply } x - \frac{y^2}{x} \text{ by } \frac{x}{y} + \frac{y}{x}.$$

$$16. \text{ Multiply } a + \frac{2a^2}{ab} \text{ by } \frac{2ab}{a^2}.$$

### CASE III.

#### 186. To Multiply an Integer by a Fraction.

$$1. \text{ Multiply the integer } a \text{ by } \frac{dx}{cy}.$$

ANALYSIS.—Changing the integer to the form of a fraction, we have  $\frac{a}{1}$  to be multiplied

by  $\frac{dx}{cy}$ , which equals  $\frac{adx}{cy}$ . Hence, the

$$a = \frac{a}{1}$$

$$\frac{a}{1} \times \frac{dx}{cy} = \frac{adx}{cy}.$$

**RULE.**—Reduce the integer to a fraction; then multiply the numerators together for the new numerator, and the denominators for the new denominator.

**NOTES.**—1. Multiplying an integer by a fraction is the same as finding a fractional part of a quantity. Thus,  $x \times \frac{3}{4}$  is the same as finding  $\frac{3}{4}$  of  $x$ , each being equal to  $\frac{3x}{4}$ . That is,

2. Three times 1 fourth of a quantity is the same as 1 fourth of 3 times that quantity.

---

186. How multiply an integer by a fraction? *Note.* To what is this operation similar?

Find the product of the following quantities:

$$2. \quad abc \times \frac{dx}{cy}.$$

$$9. \quad (x^2 - y^2) \times \frac{ac}{3(x + y)}.$$

$$3. \quad ad \times \frac{b + c}{xy}.$$

$$10. \quad (a^2 + ab) \times \frac{3c}{2(a + b)}.$$

$$4. \quad ax \times \frac{m + n}{4a}.$$

$$11. \quad (x^2 + 1) \times \frac{2ax}{3(x - 1)}.$$

$$5. \quad (a + h) \times \frac{4c}{d}.$$

$$12. \quad 2xy(a - b) \times \frac{4x}{a^2 - b^2}.$$

$$6. \quad (3a - y) \times \frac{5x}{y}.$$

$$13. \quad 3a(x - 1) \times \frac{2m}{x^2 - 1}.$$

$$7. \quad (x^2 + 1) \times \frac{5b}{x - 1}.$$

$$14. \quad (2ab + b^2) \times \frac{xy}{4a + b}.$$

$$8. \quad (1 - a^2) \times \frac{7x}{1 + a}.$$

$$15. \quad (1 - n^2) \times \frac{1}{n + 1}.$$

**187.** The principles developed in the preceding cases may be summed up in the following

#### GENERAL RULE.

*Reduce whole and mixed quantities to improper fractions, then cancel the common factors, and place the product of the numerators over the product of the denominators.*

1. Multiply  $\frac{3c - d}{20}$  by  $15x$ .

2. Multiply  $\frac{3x}{y - 1}$  by  $y^2 - 1$ .

3. Multiply  $x + \frac{2x^2}{xy}$  by  $\frac{x + y}{x^2}$ .

4. Multiply  $\frac{2ax}{a} \times \frac{3ab}{ac}$  by  $\frac{3ac}{2ab}$ .

5. Multiply  $\frac{3a^2}{10y}$  by  $\frac{5y}{9a}$ .

6. Multiply  $\frac{a^2 - b^2}{a} \times \frac{a}{a + b}$  by  $\frac{x}{a - b}$ .

7. Multiply  $\frac{m^2}{x^2 - z^2}$  by  $x + z$ .
8. Multiply  $\frac{3a^2}{y}$  by  $2y^2$ .
9. Multiply  $\frac{x + y}{x - y}$  by  $x^2 - 2xy + y^2$ .
10. Multiply  $\frac{2x + y}{40z - 10}$  by  $8z - 2$ .
11. Multiply  $x - \frac{y^2}{x}$  by  $\frac{x}{y} + \frac{y}{x}$ .
12. Multiply  $a + \frac{2a^2}{ab}$  by  $\frac{2ab}{a^2}$ .
13. Multiply  $\frac{(c + d)^2}{2a}$  by  $\frac{4a^2}{(c + d)}$ .
14. Multiply  $\frac{2x - y}{4x}$  by  $\frac{6x - 2y}{y^2 - 2xy}$ .
15. Multiply  $b + \frac{2bc}{b - c}$  by  $b - \frac{2bc}{b + c}$ .
16. Multiply  $\frac{2a}{x - y}$  by  $\frac{x^2 - y^2}{ax}$ .

## DIVISION OF FRACTIONS.

## CASE I.

**188. To Divide a Fraction by an *Integer*.**

1. If 3 oranges cost  $\frac{9a}{n}$  dollars, what will 1 cost?

ANALYSIS.—One is 1 third of 3; therefore,

OPERATION.

1 orange will cost 1 third of  $\frac{9a}{n}$  dollars, and

$$\frac{9a}{n} \div 3 = \frac{3a}{n}$$

$\frac{1}{3}$  of  $\frac{9a}{n}$  dollars is  $\frac{3a}{n}$  dollars. (Art. 167, Prin. 2.)

2. Divide  $\frac{a}{c}$  by  $m$ .

ANALYSIS.—Since we cannot divide the numerator of the fraction by  $m$ , we multiply the denominator by it.

OPERATION.

$$\frac{a}{c} \div m = \frac{a}{c \times m} = \frac{a}{cm}$$

The result is  $\frac{a}{cm}$ . For, in each of the fractions  $\frac{a}{c}$  and  $\frac{a}{cm}$ , the same number of parts is taken; but, since the unit is divided into  $m$  times as many parts in the *latter* as in the *former*, it follows that each part in the latter is only  $\frac{1}{m}$ -th of each part in the former. Hence, the

**RULE.**—*Divide the numerator by the integer.*

*Or, Multiply the denominator by it.*

**NOTE.**—If the dividend is a *mixed* quantity, it should be reduced to an improper fraction before the rule is applied. (Ex. 3.)

Divide the following quantities:

- |                                    |  |
|------------------------------------|--|
| 3. $a + \frac{bc}{x}$ by $d$ .     | <i>Ans.</i> $\frac{ax + bc}{dx}$ .               |
| 4. $x + \frac{y}{ab}$ by $xy$ .    | <i>Ans.</i> $\frac{abx + y}{abxy}$ .             |
| 5. $\frac{6x^2y}{n}$ by $3xy$ .    | 9. $\frac{a^2 + ax}{2b}$ by $a + x$ .            |
| 6. $\frac{2a^2}{3ac}$ by $b$ .     | 10. $\frac{a^3 - c^3}{b + c}$ by $a - c$ .       |
| 7. $a + \frac{ab}{c}$ by $a$ .     | 11. $\frac{x^2 + 2xy + y^2}{a + c}$ by $x + y$ . |
| 8. $ax + \frac{axy}{a}$ by $x^2$ . | 12. $\frac{x + 2y}{a - b}$ by $a + b$ .          |

## CASE II.

### 189. To Divide a Fraction by a *Fraction*.

This case embraces two classes of examples:

*First.* Those in which the fractions have a *common* denominator.

*Second.* Those in which they have *different* denominators.

188. How divide a fraction by an integer? *Note.* If the dividend is a mixed quantity, how proceed?

1. At  $\frac{3a}{m}$  dollars apiece, how many kites can a lad buy for  $\frac{12a}{m}$  dollars?

ANALYSIS.—Since these fractions have a *common denominator*, their numerators are *like* quantities, and one may be divided by the other, as integers. (Art. 177.) Now  $3a$  is contained in  $12a$ , 4 times. (Art. 110, Prin. 1.)

OPERATION.

$$\frac{12a}{m} \div \frac{3a}{m} = 4$$

Ans. 4 kites.

2. How many times is  $\frac{5ab}{x}$  contained in  $\frac{15ab}{x}$ ?

3. What is the quotient of  $\frac{35bc}{2ad}$  divided by  $\frac{7bc}{2ad}$ ?

4. It is required to divide  $\frac{a}{x}$  by  $\frac{c}{y}$ .

ANALYSIS.—Since these fractions have *different* denominators, their numerators are *unlike* quantities; consequently, one cannot be divided by the other in this form. (Art. 114, note.) We therefore reduce them to a *common denominator*; then dividing one numerator by the other, the result is the quotient.

FIRST OPERATION.

$$\frac{a}{x} = \frac{ay}{xy}, \quad \frac{c}{y} = \frac{cx}{xy},$$

$$\frac{ay}{xy} \div \frac{cx}{xy} = \frac{ay}{cx}.$$

SECOND OPERATION.

$$\frac{a}{x} \div \frac{c}{y} = \frac{a}{x} \times \frac{y}{c} = \frac{ay}{cx}, \text{ Ans.}$$

Or, more briefly, if we *invert* the divisor, and multiply the dividend by it, we have the *same combinations* and the *same result* as before. (Art. 185.) Hence, the

RULE.—*Multiply the dividend by the divisor inverted.*

Or, *Reduce the fractions to a common denominator, and divide the numerator of the dividend by that of the divisor.*

NOTES.—I. A fraction is *inverted*, when its terms are made to change places. Thus,  $\frac{a}{b}$  inverted, becomes  $\frac{b}{a}$ .

2. The object of inverting the divisor is convenience in multiplying.

3. After the divisor is inverted, the *common factors* should be *cancelled* before the multiplication is performed.

189. How divide a fraction by a fraction when they have a common denominator? When the denominators are different, how?

Divide the following fractions:

- |  |  |
|--|--|
| 5. $\frac{ab}{cd}$ by $\frac{x}{ay}$ .               | 13. $\frac{x+y}{2b^2c^2}$ by $\frac{axy}{6bc}$ . |
| 6. $\frac{3abx}{6aby}$ by $\frac{2y}{ax}$ .          | 14. $\frac{x-a}{4cd}$ by $\frac{3bc}{2d}$ .      |
| 7. $\frac{3x^2}{4}$ by $\frac{xy}{2}$ .              | 15. $\frac{2xy}{x+y}$ by $\frac{3ax}{4y}$ .      |
| 8. $\frac{xy}{x-1}$ by $\frac{xy}{2}$ .              | 16. $\frac{2x^2y^2}{a+b}$ by $\frac{3xyz}{2y}$ . |
| 9. $\frac{a-1}{x}$ by $\frac{2}{ax}$ .               | 17. $\frac{x^4}{ax^2}$ by $\frac{ax}{bx}$ .      |
| 10. $\frac{5x^2y^2}{10ab}$ by $\frac{15xy}{20abx}$ . | 18. $\frac{36ad}{5b}$ by $\frac{18ab}{10by}$ .   |
| 11. $\frac{12(x+y)}{ab}$ by $\frac{4(x+y)}{2ab}$ .   | 19. $\frac{5b}{36ad}$ by $\frac{10by}{18ab}$ .   |
| 12. $\frac{3a^2b^2}{a+b}$ by $\frac{6ab}{a}$ .       | 20. $\frac{10by}{18ab}$ by $\frac{5b}{36ad}$ .   |

### CASE III.

#### 190. To Divide an Integer by a Fraction.

1. Divide the integer  $7dc$  by  $\frac{3ad}{b}$ .

ANALYSIS.—Having reduced the integer to a fraction, and inverted the divisor, we *cancel* the *common* factor  $d$ , and proceed as in the last case. Hence, the

OPERATION.

$$7dc \div \frac{3ad}{b} =$$

$$\frac{7dc}{1} \times \frac{b}{3ad} = \frac{7bc}{3a}, \text{ Ans.}$$

RULE.—Reduce the integer to a fraction, and multiply it by the divisor inverted.

Divide the following quantities:

- |                                |                                    |
|--------------------------------|------------------------------------|
| 2. $aby \div \frac{cx}{dm}$ .  | 5. $(5x-y) \div \frac{y}{5x}$ .    |
| 3. $ax \div \frac{4a}{m+n}$ .  | 6. $(x^2+1) \div \frac{x+1}{5a}$ . |
| 4. $(a+x) \div \frac{5c}{x}$ . | 7. $(1-a^2) \div \frac{1+a}{3x}$ . |

**191. Complex Fractions** are reduced to *simple* ones, by performing the *division indicated*.

8. Reduce  $\frac{\frac{a}{b}}{\frac{3}{4}}$  to a simple fraction.

**ANALYSIS.**—The given fraction is equivalent to  $\frac{a}{b} \div \frac{3}{4}$ . Performing the division indicated, we have  $\frac{4a}{3b}$ , the simple fraction required. (Art. 189.)

$\frac{a}{b}$ , the dividend.  
 $\frac{3}{4}$ , the divisor.  
 $\frac{a}{b} \times \frac{4}{3} = \frac{4a}{3b}$ , Ans.

Reduce the following fractions to simple ones:

$$9. \frac{\frac{a^2}{b}}{cd}. \quad \text{Ans. } \frac{a^2}{bcd}.$$

$$10. \frac{\frac{a+1}{a-1}}{a+1}.$$

$$12. \frac{\frac{x^2-y^2}{a-b}}{x+y}.$$

$$11. \frac{\frac{a-b}{x+y}}{x-y}.$$

$$13. \frac{\frac{x+y}{a-b}}{a+b}.$$

**192.** The various principles developed in the preceding cases may be summed up in the following

### GENERAL RULE.

*Reduce integers and mixed quantities to improper fractions, and complex fractions to simple ones: then multiply the dividend by the divisor inverted.*

---

191. How reduce complex fractions? 192. What is the general rule for dividing fractions?

1. Divide  $\frac{15abc}{4xyz}$  by  $3bc$ .
2. Divide  $\frac{35bcd}{27y}$  by  $9xy$ .
3. Divide  $\frac{16xy}{13a}$  by  $\frac{2cd}{39a}$ .
4. Divide  $\frac{42ab}{x^2 - y^2}$  by  $\frac{14b}{x - y}$ .
5. Divide  $\frac{23xyz}{a^2 + 2ab + b^2}$  by  $\frac{17y^2}{a + b}$ .
6. Divide  $\frac{x^2 - 2xy + y^2}{3bc}$  by  $\frac{x - y}{24b^2}$ .
7. Divide  $\frac{a^4 - m^4}{a^2 - 2am + m^2}$  by  $\frac{a^2 + am}{a - m}$ .

SOLUTION.—Factoring and cancelling, we have,

$$\frac{a^4 - m^4}{a^2 - 2am + m^2} = \frac{(a^2 + m^2)(a + m)(a - m)}{(a - m)(a - m)}; \quad \frac{a^2 + am}{a - m} = \frac{a(a + m)}{a - m}. \quad \text{Then,}$$

$$\frac{(a^2 + m^2)(a + m)(a - m)}{(a - m)(a - m)} \times \frac{a - m}{a(a + m)} = \frac{a^2 + m^2}{a}, \quad \text{or } a + \frac{m^2}{a}, \text{ Ans.}$$

8. Divide  $\frac{3a}{a^2 - x^2}$  by  $\frac{3}{a - x}$ .
9. Divide  $\frac{4c^2 - 8c}{x + y}$  by  $\frac{c^2 - 4}{x + y}$ .
10. Divide  $\frac{d^2 - dx}{ac + ax}$  by  $\frac{3(c - x)}{4(d + x)}$ .
11. Divide  $\frac{2c^2}{a^8 + c^8}$  by  $\frac{c}{a + c}$ .
12. Divide  $\frac{4(a^2 - x^2)}{x}$  by  $\frac{3(a + x)}{a - x}$ .
13. Divide  $\frac{a - b}{a^2 + 2ab + b^2}$  by  $\frac{a^2 - b^2}{a + b}$ .
14. Divide  $\frac{x}{x^2 - 1}$  by  $\frac{x + 1}{x - 1}$ .
15. Divide  $\frac{bc^2 + bcd}{x + ax}$  by  $\frac{b(c + d)}{1 + a}$ . (See Appendix, p. 285.)

## CHAPTER IX.

### SIMPLE EQUATIONS.

**193.** An *Equation* is an expression of equality between two quantities. (Art. 27.)

**194.** Every equation consists of *two parts*, called the *first* and *second members*.

**195.** The *First Member* is the part on the *left* of the sign =.

The *Second Member* is the part on the *right* of the sign =.

**196.** Equations are divided into *degrees*, according to the exponent of the unknown quantity; as the first, second, third, fourth, etc.

Equations are also divided into *Simple*, *Quadratic*, *Cubic*, etc.

**197.** A *Simple Equation* is one which contains only the *first* power of the unknown quantity, and is of the *first* degree; as,  $ax = d$ .

**198.** A *Quadratic Equation* is one in which the *highest* power of the unknown quantity is a *square*, and is of the *second* degree; as,  $ax^2 + cx = d$ .

**199.** A *Cubic Equation* is one in which the *highest* power of the unknown quantity is a *cube*, and is of the *third* degree; as,  $ax^3 + bx^2 - cx = d$ .

---

193. What is an equation? 194. How many parts? 195. Which is the first member? The second? 196. How are equations divided? What other divisions? 197. What is a simple equation? 198. A quadratic? 199. Cubic?

**200.** An *Identical Equation* is one in which both members have the *same form*, or may be reduced to the same form.

Thus,  $ab - c = ab - c$ , and  $8x - 3x = 5x$ , are identical.

NOTE.—Such an equation is often called an *identity*.

**201.** The *Transformation* of an equation is changing its *form* without destroying the *equality* of its members.

NOTE.—The members of an equation will retain their *equality*, so long as they are *equally increased or diminished*. (Ax. 2-5.)

## TRANSPOSITION.

**202.** *Transposition of Terms* is changing them from one side of an equation to the other without destroying the equality of its members.

**203.** *Unknown Quantities* may be combined with *known* quantities by addition, subtraction, multiplication, or division.

NOTE.—The *object* of transposition is to obtain an equation in which the *terms* containing the *unknown* quantity shall stand on one side, and the *known* terms on the other.

**204.** To *Transpose* a Term from one Member of an Equation to the other.

1. Given  $x + b = a$ , to find the value of  $x$ .

SOLUTION.—By the problem,  $x + b = a$

Adding  $-b$  to each side (Ax. 2),  $x + b - b = a - b$

Cancelling  $(+b - b)$  (Ax. 7),  $\therefore x = a - b$

This result is the same as *changing the sign* of  $b$  from  $+$  to  $-$  in the first equation, and transposing it to the other side.

2. Given  $x - d = c$ , to find the value of  $x$ .

SOLUTION.—By the problem,  $x - d = c$

Adding  $+d$  to each side (Ax. 2),  $x - d + d = c + d$

Cancelling  $(-d + d)$ , (Ax. 7.)  $\therefore x = c + d$

200. Identical? 201. What is the transformation of an equation? *Note.* Equality.

202. What is transposition of terms? 203. How combine unknown quantities? *Note.* Object of transposition?

This result is also the same as *changing the sign* of  $d$  from  $-$  to  $+$ , and transposing it to the other side. Hence, the

**RULE.**—*Transpose the term from one member of the equation to the other, and change its sign.*

**NOTE.**—In the *first* of the preceding examples, the unknown quantity is combined with one that is known by *addition*; in the *second*, with one by *subtraction*.

3. Given  $b - c + x = a - d$ , to find  $x$ .

4. Given  $x + ab - c = a + b$ , to find  $x$ .

**205.** The *Signs* of all the terms of an equation may be changed *without destroying the equality*. For, all the terms on each side may be transposed to the other, by changing their signs.

**206.** If *all the terms* on one side are transposed to the other, *each member* will be equal to 0.

Thus, if  $x + c = d$ , it follows that  $x + c - d = 0$ .

## REDUCTION OF EQUATIONS.

**207.** The *Reduction* of an equation consists in finding the *value* of the unknown quantity which it contains.

**208.** The *Value* of an unknown quantity is the *number* which, substituted for it, will satisfy the equation. Hence, it is sometimes called the *root* of the equation.

**209.** The reduction of equations depends on the following

## PRINCIPLE.

*Both members of an equation may be increased or diminished by the same quantity without destroying the equality.*

204. How transpose a term from one member of an equation to the other?  
 205. What is the effect of changing all the signs? 206. Of transposing all the terms?  
 207. In what does the reduction of an equation consist? 208. What is the value of an unknown quantity? What sometimes called? 209. Upon what principle does the reduction of equations depend?

**210.** This principle may be illustrated by a pair of scales. If 4 balls, each weighing 1 lb., are placed in each scale, they balance each other.

Adding 2 lbs. to each scale,

$$4 + 2 = 4 + 2$$

Subtracting 2 lbs. from each,

$$4 - 2 = 4 - 2$$

Multiplying each by 2,

$$4 \times 2 = 4 \times 2$$

Dividing each by 2,

$$4 \div 2 = 4 \div 2$$



**211.** To Reduce an Equation containing One Unknown Quantity by *Transposition*.

5. Given  $2x - 3a + 7 = x + 35$ , to find  $x$ .

SOLUTION.—By the problem,  $2x - 3a + 7 = x + 35$

Transposing the terms (Art. 204),  $2x - x = 35 - 7 + 3a$

Uniting the terms, (Ax. 9), *Ans.*  $x = 28 + 3a$

Therefore,  $28 + 3a$  is the value of  $x$  required. Hence, the

**RULE.**—*Transpose the unknown quantities to one side, and the known quantities to the other, and unite the terms.*

NOTES.—1. Transposing the terms is the same, in effect, as *adding* equal quantities to, or *subtracting* them from each member; hence, it is often called reduction of equations by *addition* or *subtraction*.

2. If the unknown quantity has the sign  $-$  before it, change the signs of all the terms. (Art. 205.)

**212.** When the *same term*, having the *same sign*, is on *opposite* sides of the equation, it may be *cancelled*. (Ax. 3.)

6. Reduce  $3x + a - 6 = b - 4 + 2x$ .

7. Reduce  $x - 3 + c = 2x + a - b$ .

8. Reduce  $2y + bc - ad = y + 2m - 8$ .

9. Reduce  $3ab - y + d = -2y + 17$ .

10. Reduce  $4cd + 27 - 4x + d = 28 - 3x + 3bh$ .

11. Reduce  $b + c - 4x = 32 + b - 5x + d$ .

12. Reduce  $x + 4 - 2x - 3 = 3x + 4 + 8 - 5x$ .

210. Illustrate this principle? 211. What is the rule for reducing equations?

*Note.* To what is transposition equivalent? 212. When the *same term*, having the *same sign*, is on opposite sides, what may be done?

## CLEARING OF FRACTIONS.

**213. To Reduce an Equation containing Fractions.**

1. Given  $\frac{x}{2} + \frac{2x}{6} = 27$ , to find the value of  $x$ .

SOLUTION.—By the problem,  $\frac{x}{2} + \frac{2x}{6} = 27$   
 Multiplying each term by 6, the *l. c. m.* }  $3x + 2x = 162$   
 of the denominators (Art. 148),  
 Uniting the terms,  $5x = 162$   
 Dividing each side by the coefficient, *Ans.*  $x = 32\frac{1}{2}$

2. Given  $\frac{x}{2} + \frac{x}{3} = \frac{30}{4}$ , to find the value of  $x$ .

SOLUTION.—By the problem,  $\frac{x}{2} + \frac{x}{3} = \frac{30}{4}$   
 Mult. by 12, the *l. c. m.* of denominators,  $6x + 4x = 90$   
 Uniting terms, and dividing (Art. 211), *Ans.*  $x = 9$   
 Therefore, the value of  $x$  is 9. Hence, the

**RULE.**—*Multiply each term of the equation by the least common multiple of the denominators; then, transposing and uniting the terms, divide each member by the coefficient of the unknown quantity.*

NOTES.—1. An equation may also be *cleared of fractions*, by multiplying both sides by *each denominator* separately.

2. The *reason* that clearing an equation of fractions does not destroy the equation, is because both members are *multiplied* by the same quantity. (Ax. 4.)

3. A fraction is *multiplied* by its denominator by *cancelling* the denominator. (Art. 184, Note 1.)

4. Removing the *coefficient* of a quantity *divides* the quantity by it.

5. If any given numerator is a *multiple* of its denominator, divide the *former* by the *latter* before applying the rule.

6. The unknown quantity in the last two problems is combined with those that are known by *multiplication* and *division*. Hence, the operation is often called, reduction of equations by *multiplication* and *division*.

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213. Rule for clearing an equation of fractions? *Notes.* 1. In what other way may fractions be removed? 2. Why does not this process destroy the equation? 3. What is the effect of cancelling a denominator? 4. Effect of removing a coefficient? 5. If a numerator is a multiple of its denominator, how proceed?

3. Reduce  $\frac{3x}{5} + 12 = \frac{4x}{3} + 1$ .

4. Reduce  $\frac{2x}{3} - \frac{x}{6} = 6x - 66$ .

5. Reduce  $\frac{4x}{10} + \frac{3}{5} = 35 - x$ .

**214.** When the sign  $-$  is prefixed to a fraction and the denominator is removed, the *signs of all the terms* in the numerator must be changed from  $+$  to  $-$ , or  $-$  to  $+$ .

6. Reduce  $3x - \frac{x-2}{5} = 20$ .

SOLUTION.—By the problem,

$$3x - \frac{x-2}{5} = 20$$

Removing the denominator 5,

$$15x - x + 2 = 100$$

Uniting and transposing the terms,

$$14x = 98$$

Dividing by the coefficient,

$$\text{Ans. } x = 7$$

7. Given  $\frac{4a-5b}{x} = -\frac{3b}{d}$ , to find  $x$ .

8. Given  $3x - \frac{7x}{5} = a - \frac{2b+c}{10}$ , to find  $x$ .

9. Given  $-x + \frac{x}{3} + \frac{3x}{4} = \frac{15}{24}$ , to find  $x$ .

10. Given  $a + b + c = \frac{x}{2} + \frac{x}{4} + a + b + c \div 5$ , to find  $x$ .

**215.** The principles developed by the preceding illustrations may be summed up in the following

#### GENERAL RULE.

- I. *Clear the equation of fractions.* (Art. 213.)
- II. *Transpose and unite the terms.* (Art. 204.)
- III. *Divide both sides by the coefficient of the unknown quantity.* (Art. 213, Note 4.)

PROOF.—*For the unknown quantity substitute its value, and if it satisfies the equation, the work is right.*

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214. If sign  $-$  is prefixed to a fraction? 215. What is the general rule for simple equations? How proved?

## EXAMPLES.

1. Given  $x + \frac{x}{2} + \frac{x}{4} = 14$ , to find  $x$ .
2. Given  $\frac{x}{2} + x = \frac{7x}{10} + 40$ , to find  $x$ .
3. Given  $\frac{4x}{5} + 10 = \frac{7x}{10} + 13$ , to find  $x$ .
4. Given  $\frac{14}{x-2} + 6 = 8$ , to find  $x$ .
5. Given  $x + \frac{2x+1}{5} = 10$ , to find  $x$ .
6. Given  $2x + \frac{6x+11}{5} = 18 + \frac{9x-29}{4}$ , to find  $x$ .
7. Given  $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 78$ , to find  $x$ .
8. Given  $\frac{4x-6}{6} - 8 = \frac{x-6}{4} + 5$ , to find  $x$ .
9. Given  $x + \frac{3x-5}{2} + \frac{4x-8}{6} = 12$ , to find  $x$ .
10. Given  $2x - 16 = \frac{4x+8}{3}$ , to find  $x$ .
11. Given  $\frac{2x-8}{4} + \frac{x+32}{2} + \frac{x}{3} = 30$ , to find  $x$ .
12. Given  $\frac{x}{2} + \frac{2x}{6} = 16 + \frac{x}{6}$ , to find  $x$ .
13. Given  $\frac{3x+1}{2} - 10 = \frac{2x}{3} + \frac{x-1}{6}$ , to find  $x$ .
14. Given  $\frac{6x}{10} + \frac{5x}{6} = \frac{4x}{15} - 1\frac{4}{5}$ , to find  $x$ .
15. Given  $8x + 6\frac{1}{4} - \frac{x}{2} = 8\frac{1}{4} - \frac{2x}{7} + \frac{15x}{2}$ , to find  $x$ .

NOTE.—Sometimes there is an advantage in uniting similar terms, before clearing of fractions. Thus, uniting  $6\frac{1}{4}$  with  $8\frac{1}{4}$ ; also  $-\frac{x}{2}$  with  $\frac{15x}{2}$ , we have,  $8x = 2 - \frac{2x}{7} + 8x$ ;  $\therefore x = 7$ , *Ans.*

16. Given  $\frac{3x}{6} - 6 + \frac{2x}{6} = \frac{x}{6} + \frac{1}{2}$ , to find  $x$ .

17. Given  $\frac{4x}{5} = \frac{3x}{4} + 15 - 12$ , to find  $x$ .

18. Given  $2x - 4 = \frac{x}{2} + 2$ , to find  $x$ .

19. Given  $\frac{3x}{4} + \frac{4x}{5} - 1\frac{5}{6} = \frac{3x}{5} + 17\frac{1}{6}$ , to find  $x$ .

20. Given  $\frac{x}{a} = b + c$ , to find  $x$ .

21. Given  $\frac{ax}{n} = d$ , to find  $x$ .

22. Given  $\frac{ax}{2} + \frac{bx}{3} = c$ , to find  $x$ .

23. Given  $\frac{2ax + b}{a} = \frac{cx + d}{c}$ , to find  $x$ .

24. Given  $\frac{2c}{a} + \frac{b}{x} = \frac{c}{2} + \frac{c}{a}$ , to find  $x$ .

25. Given  $\frac{2a}{3} + \frac{4x}{5} + \frac{3b}{2} = \frac{5a}{6} + \frac{12b}{2}$ , to find  $x$ .

26. Given  $\frac{3x}{2} + \frac{2a}{3} = \frac{4b}{5} + \frac{15c}{3}$ , to find  $x$ .

27. Given  $\frac{3a + x}{x} - 5 = \frac{6}{x}$ , to find  $x$ .

28. Given  $\frac{x-1}{x+1} + 1 = \frac{1}{a}$ , to find  $x$ .

29. Given  $\frac{x}{a} + \frac{x}{c-a} = \frac{a}{c} + a$ , to find  $x$ .

30. Given  $x + b = \frac{x^2}{x+b}$ , to find  $x$ .

31. Given  $x - a = \frac{x^2 + a}{x - a}$ , to find  $x$ .

32. Given  $3\left(\frac{x+b}{4}\right) + \left(\frac{x-b}{3}\right) = 4\left(\frac{x-b}{3}\right)$ , to find  $x$ .

33. Given  $\frac{5x}{9} = x - \frac{2x-56}{3}$ , to find  $x$ .

34. Given  $\frac{3x}{4} + x - \frac{x}{2} = 25$ , to find  $x$ .
35. Given  $80 = 4x - \frac{x}{2} - \frac{x}{6}$ , to find  $x$ .
36. Given  $\frac{2x+1}{3} = 2x - \frac{x+3}{4}$ , to find  $x$ .
37. Given  $10 - 2x = \frac{3x+4}{3} - \frac{24-36}{3}$ , to find  $x$ .
38. Given  $x - 3 = 15 - \frac{x+4}{11}$ , to find  $x$ .
39. Given  $x + 2 = 3x + \frac{x+8}{4} - \frac{x+6}{3}$ , to find  $x$ .
40. Given  $\frac{3x}{4} + \frac{x-4}{2} - \frac{x-10}{2} = x - 6$ , to find  $x$ .
41. Given  $\frac{11x-1}{12} = \frac{5x-11}{4} - \frac{x-1}{10}$ , to find  $x$ .
42. Given  $\frac{4x}{5} - \frac{7x}{10} = 120$ , to find  $x$ .
43. Given  $x - 20 = -\frac{2x+1}{5}$ , to find  $x$ .
44. Given  $\frac{3x-5}{2} = \frac{4-2x}{3} + 12 - x$ , to find  $x$ .
45. Given  $\frac{1-x}{6} + 10 = \frac{2x}{3} - \frac{1-3x}{2}$ , to find  $x$ .
46. Given  $\frac{x}{a+1} - \frac{x}{a-1} = b$ , to find  $x$ .
47. Given  $\frac{x}{a-b} - \frac{2+x}{a+b} = \frac{c}{a^2-b^2}$ , to find  $x$ .
48. Given  $\frac{3a+x}{x} = 5 + \frac{6}{x}$ , to find  $x$ .
49. Given  $\frac{bx}{2} = d - \frac{bx}{3}$ , to find  $x$ .
50. Given  $8a = \frac{1-x}{1+x}$ , to find  $x$ .
51. Given  $\frac{x^2+4x+4}{x+2} = \frac{4ab}{16b}$ , to find  $x$ .

## PROBLEMS.

**216.** The *Solution* of a problem is finding a *quantity* which will satisfy its *conditions*. It consists of *two parts*:

*First.*—The *Formation* of an equation which will express the conditions of the problem in algebraic language.

*Second.*—The *Reduction* of this equation.

**217.** To Solve *Problems* in Simple Equations containing one unknown Quantity.

1. A farmer divided 52 apples among 3 boys in such a manner that B had 1 half as many as A, and C 3 fourths as many as A minus 2. How many had each?

1. FORMATION—Let

$x$  = A's number.

By the conditions,

$\frac{x}{2}$  = B's "

" "

$\frac{3x}{4} - 2$  = C's "

Therefore, by Ax. 9,

$x + \frac{x}{2} + \frac{3x}{4} - 2 = 52$ , the whole.

2. REDUCTION—

$$4x + 2x + 3x - 8 = 208$$

Transposing, etc.,

$$9x = 216$$

Removing the coefficient,

$x = 24$ , A's number.

*Ans.* A had 24, B had 12, and C had  $18 - 2 = 16$ .

From this illustration we derive the following

## GENERAL RULE.

I. Represent the unknown quantity by a letter, then state in algebraic language the operations necessary to satisfy the conditions of the problem.

II. Clear the equation of fractions; then, transposing and uniting the terms, divide each member by the coefficient of the unknown quantity. (Art. 213.)

NOTE.—A careful study of the conditions of the problem will soon enable the learner to discover the quantity to be represented by the letter, and the method of forming the equation.

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216. What is the solution of a problem? Of what does it consist? 217. What is the general rule?

2. The bill for a coat and vest is \$40; the value of the coat is 4 times that of the vest. What is the value of each?

3. A bankrupt had \$9000 to pay A, B, and C; he paid B twice as much as A, and C as much as A and B. What did each receive?

4. The whole number of hands employed in a factory was 1000; there were twice as many boys as men, and 11 times as many women as boys. How many of each?

5. Two trains start at the same time, at opposite ends of a railroad 120 miles long, one running twice as fast as the other. How far will each have run at the time of meeting?

6. A man bought equal quantities of two kinds of flour, at \$10 and \$8 a barrel. How many barrels did he buy, the whole cost being \$1200?

7. If 96 pears are divided among 3 boys, so that the second shall have 2, and the third 5, as often as the first has 1, how many will each receive?

8. A post is one-fourth of its length in the mud, one-third in the water, and 12 feet above water; what is its whole length?

9. After paying away  $\frac{1}{4}$  of my money, and then  $\frac{1}{5}$  of the remainder, I have \$72. What sum had I at first?

10. Divide \$300 between A, B, and C, so that A may have twice as much as B, and C as much as both the others.

11. At the time of marriage, a man was twice as old as his wife; but after they had lived together 18 years, his age was to hers as 3 to 2. Required their ages on the wedding day.

12. A and B invest equal amounts in trade. A gains \$1260 and B loses \$870; A's money is now double B's. What sum did each invest?

13. Required two numbers whose difference is 25, and twice their sum is 114.

14. A merchant buying goods in New York, spends the first day  $\frac{1}{3}$  of his money; the second day,  $\frac{1}{4}$ ; the third day,  $\frac{1}{5}$ ; the fourth day,  $\frac{1}{6}$ ; and he then has \$300 left. How much had he at first?

15. What number is that, from the triple of which if 17 be subtracted the remainder is 22?

16. In fencing the side of a field whose length was 450 rods, two workmen were employed, one of whom built 9 rods and the other 6 rods per day. How many days did they work?

17. Two persons, 420 miles apart, take the cars at the same time to meet each other; one travels at the rate of 40 miles an hour, and the other at the rate of 30 miles. What distance does each go?

18. Divide a line of 28 inches in length into two such parts that one may be  $\frac{3}{4}$  of the other.

19. Charles and Henry have \$200, and Charles has seven times as much money as Henry. How much has each?

20. What is the time of day, provided  $\frac{3}{4}$  of the time past midnight equals the time to noon?

21. A can plow a field in 20 days, B in 30 days, and C in 40 days. In what time can they together plow it?

22. A man sold the same number of horses, cows, and sheep; the horses at \$100, the cows at \$45, and the sheep at \$5, receiving \$4800. How many of each did he sell?

23. Divide 150 oranges among 3 boys, so that as often as the first has 2, the second shall have 5, and the third 3. How many should each receive?

24. Four geese, three turkeys, and ten chickens cost \$10; a turkey cost twice as much as a goose, and a goose 3 times as much as a chicken. What was the price of each?

25. The head of a fish is 4 inches long; its tail is 12 times as long as its head, and the body is one-half the whole length. How long is the fish?

26. Divide 100 into two parts, such that one shall be 20 more than the other.

27. Divide  $a$  into two such parts, that the greater divided by  $c$  shall be equal to the less divided by  $d$ .

28. How much money has A, if  $\frac{1}{4}$ ,  $\frac{3}{8}$ , and  $\frac{1}{8}$  of it amount to \$1222?

✓ 29. What number is that,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , and  $\frac{1}{6}$  of which are equal to 60? \*

30. A man bought beef at 25 cents a pound, and twice as much mutton at 20 cents, to the amount of \$39. How many pounds of each?

31. A says to B, "I am twice as old as you, and if I were 15 years older, I should be 3 times as old as you." What were their ages?

32. The sum of the ages of A, B, and C is 110 years; B is 3 years younger than A, and 5 years older than C. What are their ages?

33. At an election, the successful candidate had a majority of 150 votes out of 2500. What was his number of votes?

34. In a regiment containing 1200 men, there were 3 times as many cavalry as artillery less 20, and 92 more infantry than cavalry. How many of each?

35. Divide \$2000 among A, B, and C, giving A \$100 more than B, and \$200 less than C. What is the share of each?

36. A prize of \$150 is to be divided between two pupils, and one is to have  $\frac{2}{3}$  as much as the other. What are the shares?

\* When the conditions of the problem contain fractional expressions, as  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , etc., we can avoid these fractions, and greatly abridge the operation, by representing the quantity sought by such a number of  $x$ 's as can be divided by each of the denominators without a remainder. This number is easily found by taking the least common multiple of all the denominators. Thus, in problem 29,

Let  $12x =$  the number.

Then will  $6x = 1$  half.

" "  $4x = 1$  third.

" "  $3x = 1$  fourth.

" "  $2x = 1$  sixth.

Hence,  $6x + 4x + 3x + 2x = 60$

$\therefore x = 4$

Finally,  $x \times 12$  or  $12x = 48$ , the number required

37. Two horses cost \$616, and 5 times the cost of one was 6 times the cost of the other. What was the price of each?

✓ 38. What were the ages of three brothers, whose united ages were 48 years, and their birthdays 2 years apart?

39. A messenger travelling 50 miles a day had been gone 5 days, when another was sent to overtake him, travelling 65 miles a day. How many days were required?

40. What number is that to which if 75 be added,  $\frac{2}{3}$  of the sum will be 250?

✓ 41. It is required to divide 48 into two parts, which shall be to each other as 5 to 3.\*

42. What quantity is that, the half, third, and fourth of which is equal to  $a$ ?

43. A and B together bought 540 acres of land, and divided it so that A's share was to B's as 5 to 7. How many acres had each?

44. A cistern has 3 faucets; the first will empty it in 6 hours, the second in 10, and the third in 12 hours. How long will it take to empty it, if all run together?

45. Divide the number 39 into 4 parts, such that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results will be equal to each other.

46. Find a number which, if multiplied by 6, and 12 be added to the product, the sum will be 66.

47. A man bought sheep for \$94; having lost 7 of them, he sold  $\frac{1}{4}$  of the remainder at cost, receiving \$20. How many did he buy?

48. A and B have the same income; A saves  $\frac{1}{4}$  of his, but B spending \$50 a year more than A, at the end of 5 years is \$100 in debt. What is their income?

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\* When the quantities sought have a given ratio to each other, the solution may be abridged by taking such a number of  $x$ 's for the unknown quantity, as will express the ratio of the quantities to each other without fractions. Thus, taking 5 $x$  for the first part, 3 $x$  will represent the second part; then  $5x + 3x = 48$ , etc.

49. A cistern is supplied with water by one pipe and emptied by another; the former fills it in 20 minutes, the latter empties it in 15 minutes. When full, and both pipes run at the same time, how long will it take to empty it?

50. What number is that, if multiplied by  $m$  and  $n$  separately, the difference of their products shall be  $d$ ?

51. A hare is 50 leaps before a greyhound, and takes 4 leaps to the hound's 3 leaps; but 2 of the greyhound's equal 3 of the hare's leaps. How many leaps must the hound take to catch the hare?

52. What two numbers, whose difference is  $b$ , are to each other as  $a$  to  $c$ ?

53. A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

54. An express messenger travels at the rate of 13 miles in 2 hours; 12 hours later, another starts to overtake him, travelling at the rate of 26 miles in 3 hours. How long and how far must the second travel before he overtakes the first?

55. A father's age is twice that of his son; but 10 years ago it was 3 times as great. What is the age of each?

56. What number is that of which the fourth exceeds the seventh part by 30?

57. Divide \$576 among 3 persons, so that the first may have three times as much as the second, and the third one-third as much as the first and second together.

58. In the composition of a quantity of gunpowder, the nitre was 10 lbs. more than  $\frac{2}{3}$  of the whole, the sulphur  $4\frac{1}{2}$  lbs. less than  $\frac{1}{6}$  of the whole, the charcoal 2 lbs. less than  $\frac{1}{3}$  of the nitre. What was the amount of gunpowder? \*

\* The operation will be shortened by the following artifice:

Let  $42x + 48$  = the number of pounds of powder.

Then  $28x + 42$  = nitre;  $7x + 3\frac{1}{2}$  = sulphur;  $4x + 4$  = charcoal.

Hence,  $39x + 49\frac{1}{2}$  =  $42x + 48$ .

$\therefore x = \frac{1}{2}$ , and  $42x + 48$  = 69 pounds, *Ans.*

59. Divide 36 into 3 parts, such that  $\frac{1}{2}$  of the first,  $\frac{1}{3}$  of the second, and  $\frac{1}{4}$  of the third are all equal to each other.

60. Divide a line 21 inches long into two parts, such that one may be  $\frac{3}{4}$  of the other.

61. A milliner paid \$5 a month for rent, and at the end of each month added to that part of her money which was not thus spent a sum equal to 1 half of this part; at the end of the second month her original money was doubled. How much had she at first?

62. A man was hired for 60 days, on condition that for every day he worked he should receive 75 cents, and for every day he was absent he should forfeit 25 cents; at the end of the time he received \$12. How many days did he work?

63. Divide \$4200 between two persons, so that for every \$3 one received, the other shall receive \$5.

64. A father told his son that for every day he was perfect in school he would give him 15 cents; but for every day he failed he should charge him 10 cents. At the end of the term of 12 weeks, 60 school days, the boy received \$6. How many days did he fail?

65. A young man spends  $\frac{1}{3}$  of his annual income for board,  $\frac{1}{8}$  for clothing,  $\frac{1}{10}$  in charity, and saves \$318. What is his income?

66. A certain sum is divided so that A has \$30 less than  $\frac{1}{2}$ , B \$10 less than  $\frac{1}{3}$ , and C \$8 more than  $\frac{1}{4}$  of it. What does each receive, and what is the sum divided?

67. The ages of two brothers are as 2 to 3; four years hence they will be as 5 to 7. What are their ages?\*

NOTE.—To change a *proportion* into an *equation*, it is necessary to assume the truth of the following well established principle:

*If four quantities are proportional, the product of the extremes is*

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\* A strict conformity to system would require that this and similar problems should be placed after the subject of proportion; but it is convenient for the learner to be able to convert a proportion into an equation at this stage of his progress.

*equal to the product of the means.* Hence, in such cases, we have only to make the product of the extremes one side of the equation, and the product of the means the other.

Thus, let  $2x$  and  $3x$  be equal to their respective ages.

$$\text{Then} \quad 2x+4 : 3x+4 :: 5 : 7.$$

Making the product of the extremes equal to the product of the means we have,

$$14x+28 = 15x+20.$$

Transposing, uniting terms, etc.,  $x = 8$ .

$\therefore 2x = 16$ , the younger; and  $3x = 24$ , the older.

68. What two numbers are as 3 to 4, to each of which if 4 be added, the sums will be as 5 to 6?

69. The sum of two numbers is 5760, and their difference is equal to  $\frac{1}{3}$  of the greater. What are the numbers?

70. It takes a college crew which in still water can pull at the rate of 9 miles an hour, twice as long to come up the river as to go down. At what rate does the river flow?

71. One-tenth of a rod is colored red,  $\frac{1}{20}$  orange,  $\frac{1}{30}$  yellow,  $\frac{1}{40}$  green,  $\frac{1}{50}$  blue,  $\frac{1}{60}$  indigo, and the remainder, 302 inches, violet. What is its length?

72. Of a certain dynasty,  $\frac{1}{3}$  of the kings were of the same name,  $\frac{1}{4}$  of another,  $\frac{1}{5}$  of another,  $\frac{1}{7}$  of another, and there were 5 kings besides. How many were there of each name?

73. The difference of the squares of two consecutive numbers is 15. What are the numbers?

74. A deer is 80 of her own leaps before a greyhound; she takes 3 leaps for every 2 that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the deer have taken before she is caught?

75. Two steamers sailing from New York to Liverpool, a distance of 3000 miles, start from the former at the same time, one making a round trip in 20 days, the other in 25 days. How long before they will meet in New York, and how far will each have sailed?

## CHAPTER X.

### SIMULTANEOUS EQUATIONS.

#### TWO UNKNOWN QUANTITIES.

- **218. *Simultaneous\* Equations*** consist of two or more equations, each containing *two or more unknown quantities*. They are so called because they are satisfied by the same values.

Thus,  $x+y=7$  and  $5x-4y=8$  are simultaneous equations, for in each  $x=4$  and  $y=3$ .

**219. *Independent Equations*** are those which express *different conditions*, so that one cannot be reduced to the same form as the other.

Thus,  $6x-4y=14$  and  $2x+3y=22$  are independent equations. But the equations  $x+y=5$  and  $3x+3y=15$  are not *independent*, for one is directly obtained from the other. Such equations are termed *dependent*.

NOTE.—*Simultaneous* equations are usually *independent*; but *independent* equations may not be *simultaneous*; for the letters employed may have the *same* or *different* values in the respective equations.

Thus, the equations  $x+y=7$  and  $2x-2y=14$  are independent, but not simultaneous; for in one  $x=7-y$ , in the other  $x=7+y$ , etc.

**220.** Problems containing *more than one* unknown quantity must have as *many simultaneous* equations as there are unknown quantities.

If there are *more* equations than unknown quantities, some of them will be *superfluous* or *contradictory*.

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218. What are simultaneous equations? 219. Independent equations? 220. How many equations must each problem have?

\* From the Latin *simul*, at the same time.

If the number of equations be *less* than the number of unknown quantities, the problem will not admit of a *definite* answer, and is said to be *indeterminate*.

**221. Elimination\*** is combining two equations which contain two unknown quantities into a single equation, having but one unknown quantity. There are three methods of elimination, viz.: by *Comparison*, by *Substitution*, and by *Addition* or *Subtraction*.

### CASE I.

#### 222. To Eliminate an Unknown Quantity by *Comparison*.

1. Given  $x + y = 16$ , and  $x - y = 4$ , to find  $x$  and  $y$ .

SOLUTION.—By the problem,  $x + y = 16$  (1)

“ “  $x - y = 4$  (2)

Transposing the  $y$  in (1),  $x = 16 - y$  (3)


“ the  $y$  in (2),  $x = 4 + y$  (4)

By Axiom 1,  $4 + y = 16 - y$  (5)

Transposing and uniting terms,  $2y = 12$  (6)

$\therefore y = 6$


Substituting the value of  $y$  in (4),  $x = 10$

 In (5) it will be seen we have a new equation which contains only one unknown quantity. This equation is reduced in the usual way. Hence, the

**RULE.—I.** *From each equation find the value of the quantity to be eliminated in terms of the other quantities.*

**II.** *Form a new equation from these equal values, and reduce it by the preceding rules.*

**NOTE.**—This rule depends upon the axiom, that things which are equal to the same thing are equal to each other. (Ax. 1.)

 For convenience of reference, the equations are numbered (1), (2), (3), (4), etc.

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221. What is elimination? Name the methods. 222. How eliminate an unknown quantity by comparison? *Note.* Upon what principle does this rule depend?

\* From the Latin *eliminare*, to cast out.

2. Given  $x + y = 12$ , and  $x - y + 4 = 8$ , to find  $x$  and  $y$ .
3. Given  $3x + 2y = 48$ , and  $2x - 3y = 6$ , to find  $x$  and  $y$ .
4. Given  $x + y = 20$ , and  $2x + 3y = 42$ , to find  $x$  and  $y$ .
5. Given  $4x + 3y = 13$ , and  $3x + 2y = 9$ , to find  $x$  and  $y$ .
6. Given  $3x + 2y = 118$ , and  $x + 5y = 191$ , to find  $x$  and  $y$ .
7. Given  $4x + 5y = 22$ , and  $7x + 3y = 27$ , to find  $x$  and  $y$ .

## CASE II.

**223. To Eliminate an Unknown Quantity by Substitution.**

8. Given  $x + 2y = 10$ , and  $3x + 2y = 18$ , to find  $x$  and  $y$ .

SOLUTION.—By the problem,

$$x + 2y = 10 \quad (1)$$

$$\text{“ “ } 3x + 2y = 18 \quad (2)$$


$$\text{Transposing } 2y \text{ in (1), } x = 10 - 2y \quad (3)$$

$$\text{Substituting the value of } x \text{ in (2), } 30 - 4y = 18 \quad (4)$$

$$\text{Transposing and uniting terms (Art. 211), } 4y = 12 \quad (5)$$

$$\therefore y = 3$$

$$\text{Substituting the value of } y \text{ in (1), } x = 4$$

 For convenience, we first find the value of the letter which is least involved. Hence, the

RULE.—I. *From one of the equations find the value of the unknown quantity to be eliminated, in terms of the other quantities.*

II. *Substitute this value for the same quantity in the other equation, and reduce it as before.*

NOTES.—1. This method of elimination depends on Ax. 1.

2. The given equations should be cleared of fractions before commencing the elimination.

9. Given  $x + 3y = 19$ , and  $5x - 2y = 10$ , to find  $x$  and  $y$ .
10. Given  $\frac{x}{2} + \frac{y}{3} = 7$ , and  $\frac{x}{3} + \frac{y}{2} = 8$ , to find  $x$  and  $y$ .
11. Given  $2x + 3y = 28$ , and  $3x + 2y = 27$ , to find  $x$  and  $y$ .
12. Given  $4x + y = 43$ , and  $5x + 2y = 56$ , to find  $x$  and  $y$ .
13. Given  $5x + 8 = 7y$ , and  $5y + 32 = 7x$ , to find  $x$  and  $y$ .
14. Given  $4x + 5y = 22$ , and  $7x + 3y = 27$ , to find  $x$  and  $y$ .

223. How eliminate an unknown quantity by substitution? *Note.* Upon what principle does this method depend?


## CASE III.

**224. To Eliminate an Unknown Quantity by Addition or Subtraction.**15. Given  $4x + 3y = 18$ , and  $5x - 2y = 11$ , to find  $x$  and  $y$ .SOLUTION.—By the problem,  $4x + 3y = 18$  (1)" "  $5x - 2y = 11$  (2)Multiplying (1) by 2, the coef. of  $y$  in (2),  $8x + 6y = 36$  (3)Multiplying (2) by 3, the coef. of  $y$  in (1),  $15x - 6y = 33$  (4)Adding (3) and (4) cancels  $6y$ ,  $23x = 69$  (5)

$$\therefore x = 3$$

Substituting the value of  $x$  in (1),  $12 + 3y = 18$ 


$$y = 2$$

 In the preceding solution,  $y$  is eliminated by *addition*.16. Given  $6x + 5y = 28$ , and  $8x + 3y = 30$ , to find  $x$  and  $y$ .SOLUTION.—By the problem,  $6x + 5y = 28$  (1)" "  $8x + 3y = 30$  (2)Multiplying (1) by 8, the coef. of  $x$  in (2),  $48x + 40y = 224$  (3)Multiplying (2) by 6, the coef. of  $x$  in (1),  $48x + 18y = 180$  (4)Subtracting (4) from (3),  $22y = 44$ 

$$\therefore y = 2$$

Substituting the value of  $y$  in (2)  $8x + 6 = 30$ 

$$\therefore x = 3$$

 In this solution,  $x$  is eliminated by *subtraction*. Hence, the

**RULE.—I.** Select the letter to be eliminated; then multiply or divide one or both equations by such a number as will make the coefficients of this letter the same in both. (Ax. 4, 5.)

**II.** If the signs of these coefficients are alike, subtract one equation from the other; if unlike, add the two equations together. (Ax. 2, 3.)

**NOTES.—I.** The object of multiplying or dividing the equations is to equalize the coefficients of the letter to be eliminated.

**2.** If the coefficients of the letter to be eliminated are prime numbers, or *prime* to each other, multiply each equation by the coefficient of this letter in the other equation, as in Ex. 15.

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224. What is the rule for elimination by addition or subtraction? *Notes.—1.* The object of multiplying or dividing the equation? **2.** If coefficients are prime?

3. If not prime, divide the *l. c. m.* of the coefficients of the letter to be eliminated by each of these coefficients, and the respective quotients will be the multipliers of the corresponding equations. Thus, the *l. c. m.* of 6 and 8, the coefficients of  $x$  in Ex. 16, is 24; hence, the multipliers would be 3 and 4.

4. If the coefficients of the letter to be eliminated have *common factors*, the operation is shortened by *cancelling* these factors before the multiplication is performed. Thus, by cancelling the common factor 2 from 6 and 8, the coefficients of  $x$  in the last example, they become 3 and 4, and the labor of finding the *l. c. m.* is avoided.

17. Given  $3x + 4y = 29$ , and  $7x + 11y = 76$ , to find  $x$  and  $y$ .

18. Given  $9x - 4y = 8$ , and  $13x + 7y = 101$ , to find  $x$  and  $y$ .

19. Given  $3x - 7y = 7$ , and  $12x + 5y = 94$ , to find  $x$  and  $y$ .

20. Given  $3x + 2y = 118$ , and  $x + 5y = 191$ , to find  $x$  and  $y$ .

21. Given  $4x + 5y = 22$ , and  $7x + 3y = 27$ , to find  $x$  and  $y$ .

REM.—The preceding methods of elimination are applicable to all simultaneous simple equations containing two unknown quantities, and either may be employed at the pleasure of the learner.

The *first method* has the merit of *clearness*, but often gives rise to *fractions*.

The *second* is *convenient* when the coefficient of one of the unknown quantities is 1; if more than 1, it is liable to produce *fractions*.

The *third* never gives rise to *fractions*, and, in general, is the most simple and expeditious.

### EXAMPLES.

Find the values of  $x$  and  $y$  in the following equations:

1.  $2x + 3y = 23,$

$5x - 2y = 10.$

5.  $5x + 7y = 43,$

$11x + 9y = 69.$

2.  $4x + y = 34,$

$4y + x = 16.$

6.  $8x - 21y = 33,$

$6x + 35y = 177.$

3.  $30x + 40y = 270,$

$50x + 30y = 340.$

7.  $21y + 20x = 165,$

$77y - 30x = 295.$

4.  $2x + 7y = 34,$

$5x + 9y = 51.$

8.  $11x - 10y = 14,$

$5x + 7y = 41.$

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Notes.—3. If not prime, how proceed? 4. If the coefficients have common factors, how shorten the operation?

9.  $6y - 2x = 208,$

$10x - 4y = 156.$

10.  $4x + 3y = 22,$

$5x - 7y = 6.$

11.  $3x - 5y = 13,$

$2x + 7y = 81.$

12.  $5x - 7y = 33,$

$11x + 12y = 100.$

13.  $\frac{x}{5} + \frac{y}{6} = 18,$

$\frac{x}{2} - \frac{y}{4} = 21.$

14.  $16x + 17y = 500,$

$17x - 3y = 110.$

15.  $8x + y = 42,$

$2x + 4y = 18.$

16.  $2x + 4y = 20,$

$4x + 5y = 28.$

17.  $4x + 3y = 50,$

$3x - 3y = 6.$

18.  $3x + 5y = 57,$

$5x + 3y = 47.$

19.  $\frac{x}{2} + \frac{y}{3} = 7,$

$\frac{x}{3} + \frac{y}{4} = 5.$

20.  $2x + y = 50,$

$\frac{x}{6} + \frac{y}{7} = 5.$

## PROBLEMS.

1. Required two numbers whose sum is 70, and whose difference is 16.

2. A boy buys 8 lemons and 4 oranges for 56 cents; and afterwards 3 lemons and 8 oranges for 60 cents. What did he pay for each?

3. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91. How many voted for each?

4. Divide the number 75 into two such parts that three times the greater may exceed seven times the less by 15.

5. A farmer sells nine horses and seven cows for \$1200; and six horses and thirteen cows for an equal amount. What was the price of each?

6. From a company of ladies and gentlemen, 15 ladies retire; there are then left two gentlemen to each lady. After which 45 gentlemen depart, when there are left five ladies to each gentleman. How many were there of each at first?

7. Find two numbers, such that the sum of five times the first and twice the second is 19; and the difference between seven times the first and six times the second is 9.

8. Two opposing armies number together 21,110 men; and twice the number of the greater army added to three times that of the less is 52,219. How many men in each army?

9. A certain number is expressed by two digits. The sum of these digits is 11; and if 13 be added to the first digit, the sum will be three times the second. What is the number?

10. A and B possess together \$570. If A's share were three times and B's five times as great as each really is, then both would have \$2350. How much has each?

11. If 1 be added to the numerator of a fraction, its value is  $\frac{1}{3}$ ; and if 1 be added to the denominator, its value is  $\frac{1}{4}$ . What is the fraction.

12. A owes \$1200; B, \$2550. But neither has enough to pay his debts. Said A to B, Lend me  $\frac{1}{3}$  of your money, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you lend me  $\frac{1}{6}$  of yours. What sum has each?

13. Find two numbers whose difference is 14, and whose sum is 48.

14. A house and garden cost \$8500, and the price of the garden is  $\frac{5}{12}$  the price of the house. Find the price of each.

15. Divide 50 into two such parts that  $\frac{3}{4}$  of one part, added to  $\frac{5}{6}$  of the other, shall be 40.

16. Divide \$1280 between A and B, so that seven times A's share shall equal nine times B's share.

17. The ages of two men differ by 10 years; 15 years ago, the elder was twice as old as the younger. Find the age of each.

18. A man owns two horses and a saddle. If the saddle, worth \$50, be put on the first horse, the value of the two is double that of the second horse; but if the saddle be put

on the second horse, the value of the two is \$15 less than that of the first horse. Required the value of each horse.

19. A war-steamer in chase of a ship 20 miles distant, goes 8 miles while the ship sails 7. How far will each go before the steamer overtakes the ship?

20. There are two numbers, such that  $\frac{1}{2}$  the greater added to  $\frac{1}{3}$  the less is 13; and if  $\frac{1}{2}$  the less be taken from  $\frac{1}{3}$  the greater, the remainder is nothing. Find the numbers.

21. The mast of a ship is broken in a gale. One-third of the part left, added to  $\frac{1}{6}$  of the part carried away, equals 28 feet; and five times the former part diminished by 6 times the latter equals 12 feet. What was the height of the mast?

22. A lady writes a poem of half as many verses less two as she is years old; and if to the number of her years that of her verses be added, the sum is 43. How old is she? How many verses in the poem?

23. What numbers are those whose difference is 20, and the quotient of the greater divided by the less is 3?

24. A man buys oxen at \$65 and colts at \$25 per head, and spends \$720; if he had bought as many oxen as colts, and *vice versa*, he would have spent \$1440. How many of each did he purchase?

25. There is a certain number, to the sum of whose digits if you add 7, the result will be 3 times the left-hand digit; and if from the number itself you subtract 18, the digits will be inverted. Find the number.

26. A and B have jointly \$9800. A invests the sixth part of his property in business, and B the fifth part of his, and each has then the same sum remaining. What is the entire capital of each?

27. A purse holds six guineas and nineteen silver dollars. Now five guineas and four dollars fill  $\frac{1}{3}$  of it. How many will it hold of each?

28. The sum of two numbers is  $a$ , and the greater is  $n$  times the less. What are the numbers?

## THREE OR MORE UNKNOWN QUANTITIES.

**225.** The preceding methods of elimination of two unknown quantities are applicable to equations containing *three or more* unknown quantities. (Arts. 222–224.)

**226. To Solve Equations containing three or more Unknown Quantities.**

1. Given  $3x + 2y - 5z = 8$ ,  $2x + 3y + 4z = 16$ , and  $5x - 6y + 3z = 6$ , to find  $x$ ,  $y$ , and  $z$ .

SOLUTION.—By the problem,  $3x + 2y - 5z = 8$  (1)

“ “  $2x + 3y + 4z = 16$  (2)

“ “  $5x - 6y + 3z = 6$  (3)

Multiplying (1) by 2,  $6x + 4y - 10z = 16$  (4)

“ (2) by 3,  $6x + 9y + 12z = 48$  (5)

Subtracting (4) from (5),  $5y + 22z = 32$  (6)

Multiplying (2) by 5,  $10x + 15y + 20z = 80$  (7)

“ (3) by 2,  $10x - 12y + 6z = 12$  (8)

Subtracting (8) from (7),  $27y + 14z = 68$  (9)

Multiplying (6) by 27,  $135y + 594z = 864$  (10)

“ (9) by 5,  $135y + 70z = 340$  (11)

Subtracting (11) from (10),  $524z = 524$  (12)

$\therefore z = 1$

Substituting the value of  $z$  in (6),  $y = 2$

Substituting the value of  $y$  and  $z$  in (2),  $x = 3$

Ans.  $x = 3$ ,  $y = 2$ ,  $z = 1$ . Hence, the

**RULE.—I.** From the given equations eliminate one unknown quantity, by combining one equation with another.

**II.** From the resulting equations eliminate another unknown quantity in a similar manner. Continue the operation until a single equation is obtained, with but one unknown quantity, and reduce this by the preceding rules.

**NOTE.**—The letter having the *smallest* coefficients should be eliminated *first*; and if each letter is not found in all the given equations, begin with that which is in the *least number* of the equations.

226. What is the rule for solving equations having three or more unknown quantities? Which letter should be eliminated first?

Reduce the following equations:

$$\begin{aligned} 2. \quad & 5x - 3y + 2z = 28, \\ & 3x + 2y - 4z = 15, \\ & 3y + 4z - x = 24. \end{aligned}$$

$$\begin{aligned} 5. \quad & 5x + 2y + 4z = 46, \\ & 3x + 2y + z = 23, \\ & 10x + 5y + 4z = 75. \end{aligned}$$

$$\begin{aligned} 3. \quad & 2x + 5y - 3z = 4, \\ & 4x - 3y + 2z = 9, \\ & 5x + 6y - 2z = 18. \end{aligned}$$

$$\begin{aligned} 6. \quad & x + y + z = 53, \\ & x + 2y + 3z = 105, \\ & x + 3y + 4z = 134. \end{aligned}$$

$$\begin{aligned} 4. \quad & 2x + 3y - 4z = 20, \\ & x - 2y + 3z = 6, \\ & 3x - 2y + 5z = 26. \end{aligned}$$

$$\begin{aligned} 7. \quad & 3x + 4z = 57, \\ & 2y - z = 11, \\ & 5x + 3y = 65. \end{aligned}$$

$$8. \quad \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62, \quad \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47, \quad \frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38.$$

9. Required the value of  $w$ ,  $x$ ,  $y$ , and  $z$  in the following equations:

$$w + x + y + z = 14 \quad (1)$$

$$2w + x + y - z = 6 \quad (2)$$

$$2w + 3x - y + z = 14 \quad (3)$$

$$w - x + 3y + 4z = 31 \quad (4)$$

#### SOLUTION.

$$\text{Adding (1) and (2),} \quad 3w + 2x + 2y = 20 \quad (5)$$

$$\text{" (2) and (3),} \quad 4w + 4x = 20 \quad (6)$$

$$\text{Multiply (3) by 4,} \quad 8w + 12x - 4y + 4z = 56 \quad (7)$$

$$\text{• Subtract (4) from (7),} \quad 7w + 13x - 7y = 25 \quad (8)$$

$$\text{Multiply (5) by 7,} \quad 21w + 14x + 14y = 140 \quad (9)$$

$$\text{" (8) by 2,} \quad 14w + 26x - 14y = 50 \quad (10)$$

$$\text{Add (9) and (10),} \quad 35w + 40x = 190 \quad (11)$$

$$\text{Multiply (6) by 10,} \quad 40w + 40x = 200 \quad (12)$$

$$\text{Subtract (11) from (12),} \quad 5w = 10$$

$$\therefore w = 2$$

$$\text{Substituting value of } w \text{ in (6),} \quad 8 + 4x = 20$$

$$\therefore x = 3$$

Substituting value of  $w$  and  $x$  in (5), etc.,

$$y = 4, \text{ and } z = 5.$$

**227.** The solution of equations containing many unknown quantities may often be shortened by substituting a single letter for several.

10. Required the value of  $w, x, y,$   
and  $z$  in the adjoining equations.

$$\begin{cases} w+x+y=13 & (1) \\ w+x+z=17 & (2) \\ w+y+z=18 & (3) \\ x+y+z=21 & (4) \end{cases}$$

NOTE.—Substituting  $s$  for the sum of the four quantities, we have,

$$s = w+x+y+z.$$

Equation (1) contains all the letters but  $z$ ,  $s-z=13$  (5)

" (2) " " "  $y$ ,  $s-y=17$  (6)

" (3) " " "  $x$ ,  $s-x=18$  (7)

" (4) " " "  $w$ ,  $s-w=21$  (8)

Adding the last four equations together,  $\left\{ \begin{array}{l} 4s-z-y-x-w \\ \text{or } 4s-(z+y+x+w) \\ \text{or } 4s-s \end{array} \right\} = 69$  (9)

That is,  $3s=69$  (10)

$\therefore s=23$

Substituting 23 for  $s$  in each of the four equations, we have,

$$w=2, \quad x=5, \quad y=6, \quad z=10.$$

11. Required the value of  $v,$   
 $w, x, y,$  and  $z$ , in the adjoining  
equations.

$$\begin{cases} v+w+x+y=10 & (1) \\ v+w+x+z=11 & (2) \\ v+w+y+z=12 & (3) \\ v+x+y+z=13 & (4) \\ w+x+y+z=14 & (5) \end{cases}$$

NOTE.—Adding these equations,  $4v+4w+4x+4y+4z=60$  (6)

Dividing (6) by 4,  $v+w+x+y+z=15$  (7)

Subtracting each equation from (7), we have,

$$z=5, \quad y=4, \quad x=3, \quad w=2, \quad \text{and} \quad v=1.$$

12.  $w+x+z=10,$

$x+y+z=12,$

$w+x+y=9,$

$w+y+z=11.$

13.  $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$

$\frac{1}{y} + \frac{1}{z} = \frac{7}{12},$

$\frac{1}{x} + \frac{1}{z} = \frac{3}{4}.$

(See Appendix, p. 286.)

## PROBLEMS.

1. A man has 3 sons; the sum of the ages of the first and second is 27, that of the first and third is 29, and of the second and third is 32. What is the age of each?

2. A butcher bought of one man 7 calves and 13 sheep for \$205; of a second, 14 calves and 5 lambs for \$300; and of a third, 12 sheep and 20 lambs for \$140, at the same rates. What was the price of each?

3. The sum of the first and second of three numbers is 13, that of the first and third 16, and that of the second and third 19. What are the numbers?

4. In three battalions there are 1905 men:  $\frac{1}{2}$  the first with  $\frac{1}{3}$  in the second, is 60 less than in the third;  $\frac{1}{2}$  of the third with  $\frac{1}{3}$  the first, is 165 less than the second. How many are in each?

5. A grocer has three kinds of tea: 12 lbs. of the first, 13 lbs. of the second, and 14 lbs. of the third are together worth \$25; 10 lbs. of the first, 17 lbs. of the second, and 11 lbs. of the third are together worth \$24; 6 lbs. of the first, 12 lbs. of the second, and 6 lbs. of the third are together worth \$15. What is the value of a pound of each?

6. Two pipes, A and B, will fill a cistern in 70 minutes, A and C will fill it in 84 minutes, and B and C in 140 min. How long will it take each to fill the cistern?

7. Divide \$90 into 4 such parts, that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal.

8. The sum of the distances which A, B, and C have traveled is 62 miles; A's distance is equal to 4 times C's, added to twice B's; and twice A's added to 3 times B's, is equal to 17 times C's. What are the respective distances?

9. A, B, and C purchase a horse for \$100. The payment would require the whole of A's money, with half of B's; or the whole of B's with  $\frac{1}{3}$  of C's; or the whole of C's with  $\frac{1}{4}$  of A's. How much money has each?

## CHAPTER XI.

### GENERALIZATION.

**228. Generalization** is the process of finding a *formula*, or *general rule*, by which all the problems of a *class* may be solved.

**229. A Problem is generalized** when stated in general terms which embrace all examples of its class.

**230. In all General Problems** the quantities are expressed by *letters*.

1. A marketman has 75 turkeys; if his turkeys are multiplied by the number of his chickens, the result is 225. How many chickens has he?

NOTE.—This problem may be stated in the following general terms:

**231. The Product of two Factors and one of the Factors being given, to Find the other Factor.\***

SUGGESTION.—If a *product* of *two factors* is divided by *one* of them, it is evident the *quotient* must be the *other factor*. Hence, substituting *a* for the product, *b* for the given factor, we have the following

GENERAL SOLUTION.—Let  $x$  = the required factor.

By the conditions,  $x \times b$ , or  $bx = a$ , the product. Hence, the

$$\text{FORMULA.} \quad x = \frac{a}{b}.$$

Translating this Formula into common language, we have the following

RULE.—*Divide the product by the given factor; the quotient is the factor required.*

---

228. What is generalization? 229. When is a problem generalized? 230. How are quantities expressed in general problems?

\* New Practical Arithmetic, Article 93.

Generalize the next two problems:

2. A rectangular field contains 480 square rods, and the length of one side is 16 rods. What is the length of the other side?

3. Divide 576 into two such factors that one shall be 48.

4. The product of A, B, and C's ages is 61,320 years; A is 30 years, B 40. What is the age of C?

NOTE.—The items here given may be generalized as follows:

**232. The Product of three *Factors* and two of them being given, to Find the other Factor.**

SUGGESTION.—Substituting  $a$  for the product,  $b$  for one factor, and  $c$  for the other, we have the

GENERAL SOLUTION.—Let  $x$  = the required factor.

By the conditions,  $x \times b \times c$ , or  $bcx = a$ , the product.

Removing the coefficient, we have the

$$\text{FORMULA.} \quad x = \frac{a}{bc}.$$

RULE.—*Divide the given product by the product of the given factors; the quotient is the required factor.*

5. The contents of a rectangular block of marble are 504 cubic feet; its length is 9 feet, and its breadth 8 feet. What is its height?

6. The product of 3 numbers is 62,730, and two of its factors are 41 and 45. Required the other factor.

7. The amount paid for two horses was \$392, and the difference in their prices was \$18. What was the price of each?

NOTE.—From the items given, this problem may be generalized as follows:

---

231. When the product of two factors and one of the factors are given, how find the other factor? 232. When the product of three factors and two of them are given, how find the other factor?

### 233. The Sum and Difference of two Quantities being given, to Find the Quantities.

SUGGESTION.—Since the *sum* of two quantities equals the greater *plus* the less; and the less *plus* the difference equals the greater; it follows that the *sum plus* the *difference* equals *twice* the greater. Substituting  $s$  for the sum,  $d$  for the difference,  $g$  for the greater, and  $l$  for the less, we have the following

GENERAL SOLUTION.—Let  
and

$g$  = the greater number,  
 $l$  = " less "

Adding,  $g + l = s$ , the sum.

Subtracting,  $g - l = d$ , the difference.


Adding sum and difference,  $2g = s + d$

Removing coefficient,  $g = \frac{s + d}{2}$ , greater.

Subtracting difference from sum,  $2l = s - d$

Removing coefficient,  $l = \frac{s - d}{2}$ , less. Hence, the

$$\text{FORMULAS. } \left\{ \begin{array}{l} g = \frac{s + d}{2} \\ l = \frac{s - d}{2} \end{array} \right.$$

 This problem may be solved by *one* unknown quantity.

### FORMATION OF RULES.

234. Many of the more important rules of Arithmetic are formed by translating *Algebraic Formulas* into common language. Thus, from the translation of the two preceding formulas into common language, we have, for all problems of this class, the following general

RULE.—I. *To find the greater, add half the sum to half the difference.*

II. *To find the less, subtract half the difference from half the sum.*

8. Divide \$1575 between A and B in such a manner that A may have \$347 more than B. What will each receive?

233. When the sum and difference of two quantities are given, how find the quantities? 234. Give the rule derived from the last two formulas.

9. At an election there were 2150 votes cast for two persons; the majority of the successful candidate was 346. How many votes did each receive?

10. If B can do a piece of work in 8 days, and C in 12 days, how long will it take both to do it?

NOTE.—Regarding the work to be done as a *unit* or 1, the problem may be thus generalized:

**235.** The Time being given in which each of two Forces can produce a given Result, to Find the Time required by the united Forces to produce it.

SUGGESTION.—Since B can do the work in 8 days, he can do 1 eighth of it in 1 day, and C can do 1 twelfth of it in 1 day. Substituting  $a$  for 8 days, and  $b$  for 12 days, we have the

GENERAL SOLUTION.—Let  $x$  = the time required.

Dividing  $x$  by  $a$ , we have  $\frac{x}{a}$  = part done by B.

“  $x$  by  $b$ , we have,  $\frac{x}{b}$  = “ “ C.

By Axiom 9,  $\frac{x}{a} + \frac{x}{b} = 1$ , the work done.

Clearing of fractions,  $bx + ax = ab$

Uniting the terms,  $(a+b)x = ab$ , B and C's time.

Removing the coefficient, we have the

$$\text{FORMULA.} \quad x = \frac{ab}{a+b}.$$

**RULE.**—*Divide the product of the numbers denoting the time required by each force, by the sum of these numbers; the quotient is the time required by the united forces.*

11. A cistern has two pipes; the first will fill it in 9 hours, the second in 15 hours. In what time will both fill it, running together?

---

235. The time being given in which two or more forces can produce a result, how find the time required for the united forces to produce it?

12. A can plant a field in 40 hours, and B can plant it in 50 hours. How long will it take both to plant it, if they work together?

**236.** In *Generalizing Problems* relating to Percentage, there is an advantage in representing the quantities, whether known or unknown, by the *initials* of the elements or factors which enter into the calculations.

NOTE.—The *elements* or *factors* in percentage are,

1st. The *Base*, or number on which *percentage* is calculated.

2d. The *Rate per cent*, which shows how many hundredths of the base are taken.

3d. The *Percentage*, or portion of the base indicated by the rate.

4th. The *Amount*, or the base *plus* or *minus* the percentage. Thus,

Let  $b$  = the base.  $p$  = the percentage.

$r$  = the rate per cent.  $a$  = the amount.

13. A man bought a lot of goods for \$748, and sold them at 9 per cent above cost. How much did he make?

NOTE.—The items in this problem may be generalized as follows:

**237.** The Base and Rate being given, to Find the *Percentage*.\*

SUGGESTION.—*Per cent* signifies *hundredths*; hence, any given per cent of a quantity denotes so many *hundredths* of that quantity. But finding a *fractional part* is the same as *multiplying* the quantity by the given fraction. Substituting  $b$  for the *cost* or *base*, and  $r$  for the *number* denoting the *rate per cent*, we have the

GENERAL SOLUTION.—Let  $p$  = the percentage.

Multiplying the base by the rate,  $br = p$ . Hence, the

$$\text{FORMULA.} \quad p = br.$$

RULE.—*Multiply the base by the rate per cent; the product is the percentage.* Hence,

NOTE.—*Percentage* is a *product*, the *factors* of which are the *base* and *rate*.

---

236. Note. What are the elements or factors in percentage? 237. When base and rate are given, how find the percentage?

\* New Practical Arithmetic, Arts. 336—340.

14. The population of a certain city in 1870 was 45,385; in 1875 it was found to have increased 20 per cent. What was the percentage of increase?

15. Find  $37\frac{1}{2}$  per cent on \$2763.

16. A western farmer raised 1587 bushels of wheat, and sold 37 per cent of it. How many bushels did he sell?

17. A teacher's salary of \$2700 a year was increased \$336. What per cent was the increase?

NOTE.—The data of this problem may be generalized as follows :

### 238. The Base and Percentage being given, to Find the *Rate*.

SUGGESTION.—*Percentage*, we have seen, is a *product*, and the *base* is one of its factors (Art. 237, note); therefore, we have the *product* and *one* factor given, to find the other factor. (Art. 231.) Substituting *p* for the *product* or *percentage*, and *b* for the *salary* or *base*, we have the

GENERAL SOLUTION.—Let  $r$  = the required rate.

Then (Art. 231),  $p + b = r$ . Hence, the

$$\text{FORMULA.} \quad r = \frac{p}{b}.$$

RULE.—*Divide the percentage by the base; the quotient is the rate.*

18. From a hogshead of molasses, 25.2 gallons leaked out. What per cent was the leakage?

19. A steamship having 485 passengers was wrecked, and 291 of them lost. What per cent were lost?

20. A man gained \$750 by a speculation, which was 25 per cent. of the money invested. What sum did he invest?

NOTE.—The particular statement of this problem may be transformed into the following general proposition :

---

238. When the base and percentage are given, how find the rate?

**239. The Percentage and Rate being given, to Find the *Base*.**

**SUGGESTION.**—We have the *product* and *one* of its factors given, to find the *other* factor. (Art. 237, *note*.) Substituting  $p$  for the *percentage*, and  $r$  for the *rate per cent* gained, we have the

**GENERAL SOLUTION.**—Let  $b$  = the base.

Then (Art. 237),  $p+r = b$ . Hence, the

$$\text{FORMULA.} \quad b = \frac{p}{r}.$$

**RULE.**—*Divide the percentage by the rate, and the quotient is the base.*

21. A paid a tax of \$750, which was 2 per cent of his property. How much was he worth?

22. A merchant saves 8 per cent of his net income, and lays up \$2500 a year. What is his income?

23. At the commencement of business, B and C were each worth \$2500. The first year B added 8 per cent to his capital, and C lost 8 per cent of his. What amount was each then worth?

**NOTE.**—The items of this problem may be generalized thus:

**240. The Base and Rate being given, to Find the *Amount*.**

**SUGGESTION.**—Since B laid up 8 per cent., he was worth his original stock *plus* 8 per cent of it. But his stock was 100 per cent, or *once* itself; and 100 per cent. + .08 = 108 per cent or 1.08 times his stock.

Again, since C lost 8 per cent, he was worth his original stock *minus* 8 per cent of it. Now 100 per cent *minus* 8 per cent equals 100 per cent — .08 = 92 per cent, or .92 times his capital. Substituting  $b$  for the base or capital of each, and  $r$  for the number denoting the *rate per cent* of the gain or loss, we have the

**GENERAL SOLUTION.**—Let  $a$  = the amount.

Then will  $b(1+r) = a$ , B's amount.

And  $b(1-r) = a$ , C's amount.

Combining these two results, we have the

$$\text{FORMULA.} \quad a = b(1 \pm r).$$

**RULE.**—*Multiply the base by  $1 \pm$  the rate, as the case may require, and the result will be the amount.*

---

239. When percentage and rate are given, how find the base? 240. How find the amount when the base and rate are given?

NOTE.—When, from the nature of the problem, the *amount* is to be greater than the base, the multiplier is 1 *plus* the rate; when *less*, the multiplier is 1 *minus* the rate.

24. A man bought a flock of sheep for \$4500, and sold it 25 per cent above the cost. What amount did he get for it?

25. A man owned 2750 acres of land, and sold 33 per cent of it. What amount did he have left?

**241.** The elements or factors which enter into computations of interest are the *principal*, *rate*, *time*, *interest*, and *amount*. Thus,

Let  $p$  = the principal, or money lent.

“  $r$  = the interest of \$1 for 1 year, at the given rate.

“  $t$  = the time in years.

“  $i$  = the interest, or the percentage.

“  $a$  = the amount, or the *sum* of principal and interest.

26. What is the interest of \$465 for 2 years, at 6 per cent?

NOTE.—The data of this problem may be stated in the following general proposition :

**242.** The Principal, the Rate, and Time being given, to Find the *Interest*.

GENERAL SOLUTION.—Since  $r$  is the interest of \$1 for 1 year,  $p \times r$  must be the interest of  $p$  dollars for 1 year; therefore,  $pr \times t$  must be the interest of  $p$  dollars for  $t$  years. Hence, the

FORMULA.  $i = prt.$

RULE.—*Multiply the principal by the interest of \$1 for the given time, and the result is the interest.*

27. What is the interest of \$1586 for 1 yr. and 6 m., at 8 per cent?

28. What is the int. of \$3580 for 5 years, at 7 per cent?

29. What is the amt. of \$364 for 3 years, at 5 per cent?

NOTE.—This problem may be stated in the following general terms:

*Note.* When the amount is greater or less than the base, what is the multiplier?

241. What are the elements or factors which enter into computations of interest?

242. When the principal, rate, and time are given, how find the interest?

**243. The Principal, Rate, and Time being given, to Find the Amount.**

GENERAL SOLUTION.—Reasoning as in the preceding article,  
the interest =  $p r t$ .

But the *amount* is the *sum* of the principal and interest.

$$\therefore p + p r t = a. \text{ Hence, the}$$

$$\text{FORMULA.} \quad a = p + p r t.$$

RULE.—*Add the interest to the principal, and the result is the amount.*

30. Find the amount of \$4375 for 2 years and 6 months, at 8 per cent.

31. Find the amt. of \$2863.60 for 5 years, at 7 per cent.

**244. The Relation between the four elements in the**

$$\text{FORMULA,} \quad a = p + p r t,$$

is such, that if any *three* of them are given, the *fourth* may be readily found. (Art. 243.)

**245. The Amount, the Rate, and Time being given, to Find the Principal.**

Transposing the members and factoring, we have the

$$\text{FORMULA.} \quad p = \frac{a}{1 + r t}.$$

32. What principal will amount to \$1500 in 2 years, at 6 per cent?

33. What sum must be invested at 7 per cent to amount to \$300 in 5 years?

**246. The Amount, the Principal, and the Rate being given, to Find the Time.**

Transposing  $p$  and dividing by  $p r$ , (Art. 244), we have the

$$\text{FORMULA.} \quad t = \frac{a - p}{p r}.$$

34. In what time will \$3500, at 6 per cent, yield \$525 interest?

243. The amount? 244. What is the relation between the four elements in the preceding formula. 245. When the amount, rate, and time are given, state the formula. 246. When the amount, principal, and rate are given, state the formula.

**247. The Hour and Minute Hands of a Clock being together at 12 M., to Find the Time of their *Conjunction* between any two *Subsequent* Hours.**

35. The hour and minute hands of a clock are exactly together at 12 o'clock. It is required to find how long before they will be together again.

ANALYSIS.—The distance around the dial of a clock is 12 hour spaces. When the hour-hand arrives at I, the minute-hand has passed 12 hour spaces, and made an entire circuit. But since the hour-hand has moved over one space, the minute-hand has gained only 11 spaces. Now, if it takes the minute-hand 1 hour to gain 11 spaces, to gain 1 space will take  $\frac{1}{11}$  of an hour, and to gain 12 spaces it will take 12 times as long, and 12 times  $\frac{1}{11}$  hr. =  $\frac{12}{11}$  hr. =  $1\frac{1}{11}$  hour. Or,

Let  $x$  = the time of their conjunction.

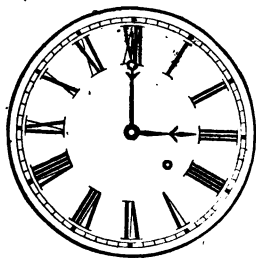
Then 11 spaces : 12 spaces :: 1 hour :  $x$  hours.

Multiplying extremes, etc.,  $11x = 12$

Removing coefficient,  $x = 1\frac{1}{11}$  hr., or 1 hr.  $5\frac{5}{11}$  min.

36. When will the hour and minute hand be in conjunction next after 3 o'clock?

SUGGESTION.—Substituting  $a$  for  $1\frac{1}{11}$  hr., the time it takes the minute-hand to gain 12 spaces,  $h$  for the given number of hours past 12 o'clock,  $t$  for the time of conjunction, we have the following



GENERAL SOLUTION.  $a \times h = t$ , the time required. Hence, the

FORMULA.  $t = ah$ .

RULE.—Multiply the time required to gain 12 spaces by the given hour past 12 o'clock; the product will be the time of conjunction.

37. At what time after 6 o'clock will the hour and minute hand be in conjunction?

38. At what time between 9 and 10 o'clock?

(See Appendix, p. 287.)

247. What is the formula for finding when the hands of a clock will be in conjunction? Translate this into a rule.

## CHAPTER XII.

### INVOLUTION.\*

**248.** *Involution* is finding a power of a quantity.

**249.** A *Power* is the product of two or more equal factors.

Thus,  $3 \times 3 = 9$ ;  $a \times a \times a = a^3$ ; 9 and  $a^3$  are powers.

**250.** Powers are divided into *different degrees*; as *first*, *second*, *third*, *fourth*, etc., the name corresponding with the *number of times* the quantity is taken as a *factor* to produce the power.

**251.** The *First Power* is the *quantity* itself.

The *Second Power* is the product of *two* equal factors, and is called a *square*.

The *Third Power* is the product of *three* equal factors, and is called a *cube*, etc.

NOTE.—The quantity *called* the *first power* is, strictly speaking, not a power, but a *root*. Thus,  $a^1$  or  $a$ , is not the product of any *two equal* factors, but is a quantity or *root* from which its *powers* arise.

**252.** The *Index* or *Exponent* † of a power is a *figure* or *letter* placed at the right, above the quantity. Its object is to show *how many times* the quantity is taken as a *factor* to produce the power.

Thus,  $a^1 = a$ , and is called the *first power*.

$a^2 = a \times a$ , the *second power*, or *square*.

$a^3 = a \times a \times a$ , the *third power*, or *cube*.

$a^4 = a \times a \times a \times a$ , the *fourth power*, etc.

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248. What is involution? 249. A power? 250. How divided? 251. The first power? Second power? Third? 252. What is the index or exponent? Its object?

\* *Involution*, from the Latin *involvere*, to roll up.

† *Index* (plural, *indices*), Latin *indicāre*, to indicate.

*Exponent*, from the Latin *exponere*, to set forth.

NOTES.—1. The index of the *first* power being 1, is commonly omitted.

2. The expression  $a^4$  is read “ $a$  fourth,” “the fourth power of  $a$ ,” or “ $a$  raised to the fourth power;”  $x^n$  is read, “ $x$  *nth*,” or “the *nth* power of  $x$ .”

Read the following:  $a^5$ ,  $c^4$ ,  $x^7$ ,  $y^{10}$ ,  $z^{15}$ ,  $b^m$ ,  $d^n$ .

**253. Powers** are also divided into *direct* and *reciprocal*.

**254. Direct Powers** are those which arise from the continued multiplication of a quantity into itself.

Thus, the continued multiplication of  $a$  into itself gives the series,  
 $a$ ,  $a^2$ ,  $a^3$ ,  $a^4$ ,  $a^5$ ,  $a^6$ , etc.

**255. Reciprocal Powers** are those which arise from the continued division of a unit by the *direct powers* of that quantity. (Art. 55.)

Thus, the continued division of a *unit* by the direct powers of  $a$  gives the series,

$$\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5}, \frac{1}{a^6}, \text{ etc.}$$

**256. Reciprocal Powers** are commonly denoted by prefixing the sign — to the *exponents* of direct powers of the *same* degree.

Thus,  $\frac{1}{a} = a^{-1}$ ,  $\frac{1}{a^2} = a^{-2}$ ,  $\frac{1}{a^3} = a^{-3}$ ,  $\frac{1}{a^4} = a^{-4}$ , etc.

**257.** The difference in the notation of *direct* and *reciprocal* powers may be seen from the following series:

(1.)  $a^5$ ,  $a^4$ ,  $a^3$ ,  $a^2$ ,  $a^1$ , 1,  $\frac{1}{a}$ ,  $\frac{1}{a^2}$ ,  $\frac{1}{a^3}$ ,  $\frac{1}{a^4}$ ,  $\frac{1}{a^5}$ , etc.

(2.)  $a^5$ ,  $a^4$ ,  $a^3$ ,  $a^2$ ,  $a^1$ ,  $a^0$ ,  $a^{-1}$ ,  $a^{-2}$ ,  $a^{-3}$ ,  $a^{-4}$ ,  $a^{-5}$ , etc.

NOTE.—The *first half* of each of the above expressions is a series of *direct* powers; the *last half*, a series of *reciprocal* powers.

**258. Negative Exponents** are the same as the exponents of *direct* powers, with the sign — prefixed to them.

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*Note.* The index 1? 253. How else are powers divided? 254. Direct powers? 255. Reciprocal? 256. How is a reciprocal power denoted?

NOTES.—1. This *notation* of reciprocal powers is derived from the *continued division* of a series of *direct* powers by their *root*; that is, by subtracting 1 from the successive exponents. (Art. 113.)

2. The use of negative exponents in expressing reciprocal powers avoids fractions, and therefore is convenient in calculations.

3. *Direct powers* are often called *positive*, and *reciprocal powers*, *negative*. But the student must not confound the quantities whose *exponents* have the sign + or —, with those whose *coefficients* have the sign + or —. This ambiguity will be avoided, by applying the term *direct*, to powers with *positive* exponents, and *reciprocal*, to those with *negative* exponents.

**259.** The *Zero Power* of a quantity is one whose exponent is 0; as,  $a^0$ ; read, “the zero power of  $a$ .”

Every quantity with the index 0, is equal to a unit or 1.

For,  $\frac{a^n}{a^n} = a^{n-n} = a^0$  (Art. 113); but  $\frac{a^n}{a^n} = 1$ ; hence,  $a^0 = 1$ .

## SIGNS OF POWERS.

**260.** When a quantity is *positive*, all its *powers* are *positive*.

Thus,  $a \times a = a^2$ ;  $a \times a \times a = a^3$ , etc.

When a quantity is *negative*, its *even* powers are *positive*, and its *odd* powers *negative*.

Thus,  $-a \times -a = a^2$ ;  $-a \times -a \times -a = -a^3$ , etc.

## FORMATION OF POWERS.

**261.** *All Powers* of a quantity may be formed by multiplying the quantity into itself. (Art. 249.)

**262.** To Raise a *Monomial* to any Required Power.

The process of involving a quantity which consists of several factors depends upon the following

259. What is the zero power? To what is a quantity of the zero power equal?

260. Rule for the signs?

## PRINCIPLES.

1°. *The power of the product of two or more factors is equal to the product of their powers.*

2°. *The product is the same, in whatever order the factors are taken.* (Art. 87, Prin. 3.)

1. Given  $3ab^2$  to be raised to the third power.

SOLUTION.

$$(3ab^2)^3 = 3ab^2 \times 3ab^2 \times 3ab^2 \text{ (Art. 261),}$$

$$\text{or, } 3 \times 3 \times 3 \times a \times a \times a \times b^2 \times b^2 \times b^2 \text{ (Prin. 2),}$$

$$\therefore (3ab^2)^3 = 27a^3b^6, \text{ Ans.}$$

Involving each of these factors separately, we have,  $(3)^3 = 27$ ;  $(a)^3 = a^3$ ; and  $(b^2)^3 = b^6$ ; and  $27 \times a^3 \times b^6 = 27a^3b^6$ , Ans. Hence, the

**RULE.**—*Raise the coefficient to the power required, and multiply the index of each letter by the index of the power, prefixing the proper sign to the result.* (Art. 90.)

**NOTES.**—1. A single letter is involved by giving it the index of the required power.

2. A quantity which is already a power is involved by multiplying its index by the index of the required power.

3. The learner must observe the distinction between an *index* and a *coefficient*. The latter is simply a multiplier, the former shows how many times the quantity is taken as a factor.

4. This rule is applicable both to positive and negative exponents.

2. What is the square of  $abc$ ?
3. What is the square of  $-abc$ ?
4. What is the cube of  $xyz$ ?
5. What is the fifth power of  $abc$ ?
6. What is the fourth power of  $2x^2y$ ?
7. What is the third power of  $6a^3b^2$ ?
8. What is the fourth power of  $5a^3b^2c$ ?
9. What is the sixth power of  $2a^2bc^2$ ?
10. What is the eighth power of  $abcd$ ?
11. What is the  $n$ th power of  $xyz$ ?

---

262. How raise a monomial to any power? *Note.* A single letter? A quantity already a power? Distinction between index and coefficient?

12. Find the fifth power of  $(a + b)^2$ .
13. Find the second power of  $(a + b)^n$ .
14. Find the  $n$ th power of  $(x - y)^m$ .
15. Find the  $n$ th power of  $(x + y)^2$ .
16. Find the second power of  $(a^3 + b^3)$ .
17. Find the third power of  $(a^3b^2h^4)$ .

**263. To Involve a Fraction to any required Power.**

18. What is the square of  $\frac{2x^2y}{3x^3y^2}$ ?

SOLUTION.  $\left(\frac{2x^2y}{3x^3y^2}\right)^2 = \frac{2x^2y}{3x^3y^2} \times \frac{2x^2y}{3x^3y^2} = \frac{4x^4y^2}{9x^6y^4}$ . Hence, the

**RULE.**—*Raise both the numerator and denominator to the required power.*

19. Find the cube of  $\frac{3ab^2}{2a}$ .
20. Find the fourth power of  $\frac{2a^2bc^3}{x^2y^3}$ .
21. Find the square of  $\frac{7a^n b^2}{3a^2 b^n}$ .
22. Find the  $m$ th power of  $\frac{2}{a}$ .
23. Find the  $n$ th power of  $\frac{a^m b^n}{xy^n}$ .

**264.** A *compound quantity* consisting of two or more terms, connected by + or —, is involved by *actual multiplication* of its several parts.

24. Find the square of  $3a + b^2$ . *Ans.*  $9a^2 + 6ab^2 + b^4$ .
25. What is the square of  $a + b + c$ ?  
*Ans.*  $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$ .
26. What is the cube of  $x + 2y + z$ ?

**265.** It is sometimes sufficient to express the power of a compound quantity by exponents.

Thus, the square of  $a + b = (a + b)^2$ ; the  $n$ th power of  $ab + c + 3d^2 = (ab + c + 3d^2)^n$ .

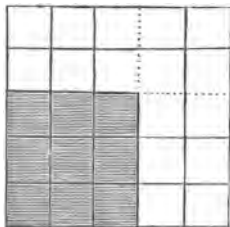
## FORMATION OF BINOMIAL SQUARES.

**266.** To Find the *Square of a Binomial* in the Terms of its Parts.

1. Given two numbers, 3 and 2, to find the square of their sum in the terms of its parts.

ILLUSTRATION.—Let the shaded part of the diagram represent the square of 3;—each side being divided into 3 inches, its contents are equal to  $3 \times 3$ , or 9 sq. in.

To preserve the form of the square, it is plain equal additions must be made to *two adjacent sides*; for, if made on *one side*, or on *opposite sides*, the figure will no longer be a square.



Since 5 is 2 more than 3, it follows that *two rows* of 3 squares each must be added at the top, and 2 rows on one of the adjacent sides, to make its *length* and *breadth* each equal to 5. Now 2 into 3 plus 2 into 3 are 12 squares, or *twice* the product of the two parts 2 and 3.

But the diagram wants two times 2 small squares, to fill the corner on the right, and 2 times 2, or 4, is the square of the second part. We have then 9 (the square of the first part), 12 (twice the product of the two parts 3 and 2), and 4 (the square of the second part). Therefore,  $(3+2)^2 = 3^2 + 2 \times (3 \times 2) + 2^2$ .

2. Required the square of  $x+y$ . *Ans.*  $x^2 + 2 \times xy + y^2$ .

Hence, universally,

*The square of the sum of two quantities is equal to the square of the first, plus twice their product, plus the square of the second.*

NOTE.—The *square of a binomial* always has *three terms*, and consequently is a *trinomial*. Hence,

*No binomial can be a perfect square.* (Art. 101.)

---

266. To what is the square of the sum of two quantities equal? How illustrate the square of the sum of two quantities in the terms of its parts?

**267.** *All Binomials* may be raised to any required power by continued multiplication. But when the exponent of the power is large, the operation is greatly abridged by means of the *Binomial Theorem*.\*

**268.** The *Binomial Theorem* is a *general formula* by which any power of a binomial may be found without recourse to continued multiplication.

To illustrate this theorem, let us raise the binomials  $a+b$  and  $a-b$  to the second, third, fourth, and fifth powers, by continued multiplications :

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

$$(a - b)^2 = a^2 - 2ab + b^2.$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

**269.** Analyzing these operations, the learner will discover the following *laws* which govern the *expansion of binomials* :

1. *The number of terms* in any power is *one more* than the *index* of the power.

2. The *index* of the *first* term or *leading* letter is the *index* of the required power, which *decreases* regularly by 1 through the other terms.

The *index* of the *following* letter begins with 1 in the second term, and *increases* by 1 through the other terms.

3. *The sum of the indices* is the *same* in each term, and is *equal* to the *index* of the power.

---

268. What is the Binomial Theorem? 269. What is the law respecting the number of terms in a power? The indices of each quantity? The sum of the indices in each term?

\* This method was discovered by Sir Isaac Newton, in 1666.

4. *The coefficient of the first and last term of every power is 1; of the second and next to the last, it is the index of the power; and, universally, the coefficients of any two terms equidistant from the extremes, are equal to each other.*

Again, the coefficients regularly *increase* in the *first half* of the terms, and *decrease* at the same rate in the *last half*.

5. *The signs follow the same rule as in multiplication*

**270.** The preceding principles may be summed up in the following

### GENERAL RULE.

**I. INDICES.**—*Give the first term or leading letter the index of the required power, and diminish it regularly by 1 through the other terms.*

*The index of the following letter in the second term is 1, and increases regularly by 1 through the other terms.*

**II. COEFFICIENTS.**—*The coefficient of the first term is 1.*

*To the second term give the index of the power; and, universally, multiplying the coefficient of any term by the index of the leading letter in that term, and dividing the product by the index of the following letter increased by 1, the result will be the coefficient of the succeeding term.*

**III. SIGNS.**—*If both terms are positive, make all the terms positive; if the second term is negative, make all the odd terms, counting from the left, positive, and all the even terms negative.*

### THE BINOMIAL FORMULA.

$$(a + b)^n = a^n + n \times a^{n-1} b + n \times \frac{n-1}{2} a^{n-2} b^2, \text{ etc.}$$

**NOTE.**—The preceding rule is based upon the supposition that the index is a *positive whole number*; but it is equally true when the index is either *positive or negative, integral or fractional*.

---

The coefficients of the first and last terms? The law of the signs? 270. What is the general rule?

Expand the following binomials:

1.  $(a + b)^4$ .

6.  $(y + z)^{10}$ .

2.  $(a - b)^5$ .

7.  $(a - b)^9$ .

3.  $(c + d)^7$ .

8.  $(m + n)^{11}$ .

4.  $(x + y)^6$ .

9.  $(x - y)^{12}$ .

5.  $(x - y)^7$ .

10.  $(a + b)^n$ .

**271.** When the *terms* of a binomial have *coefficients* or *exponents*, the operation may be shortened by substituting for them single letters of the first power. After the operation is completed, the value of the terms must be restored.

11. Required the fifth power of  $x^2 + 3y^2$ .

SOLUTION.—Substitute  $a$  for  $x^2$ , and  $b$  for  $3y^2$ ; then

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Restoring the values of  $a$  and  $b$ ,

$$(x^2 + 3y^2)^5 = x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}.$$

12. Expand  $(x^2 - 3b)^4$ .

$$\text{Ans. } x^8 - 12x^6b + 54x^4b^2 - 108x^2b^3 + 81b^4.$$

**272.** Every power of 1 is 1, and when a factor it has no effect upon the quantity with which it is connected. (Art. 94, note.) Hence, when one of the terms of a binomial is 1, it is commonly omitted in the required power, except in the *first* and *last* terms.

NOTE.—In finding the exponents of such binomials, it is only necessary to observe that the *sum* of the two exponents in each term is *equal* to the index of the power.

13. Expand  $(x + 1)^3$ .

15. Expand  $(1 - a)^5$ .

14. Expand  $(b - 1)^4$ .

16. Expand  $(1 + a)^n$ .

(See Appendix, p. 287.)

**273.** A *Polynomial* may be raised to any power by actual multiplication, taking the given quantity as a factor as many times as indicated by the *exponent* of the required power. But the operation may often be shortened by reducing the several terms to two, by substitution, and then applying the Binomial Formula.

17. Required the cube of  $x + y + z$ .

SOLUTION.—Substituting  $a$  for  $(y+z)$ , we have  $x+(y+z) = x+a$ .

By formula,  $(x+a)^3 = x^3 + 3x^2a + 3\tilde{x}a^2 + a^3$ .

Restoring the value of  $a$ ,

$$(x+y+z)^3 = x^3 + 3x^2(y+z) + 3x(y+z)^2 + (y+z)^3.$$

**274.** To *Square* a Polynomial without Recourse to Multiplication.

18. Required the square of  $a + b + c$ .

SOLUTION.—By actual multiplication, we have,

$$(a+b+c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2.$$

Or, changing the order of terms,

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Or, factoring, we have,  $a^2 + 2a(b+c) + b^2 + 2bc + c^2$ .

19. Required the square of  $a + b + c + d$ .

SOLUTION.—By actual multiplication, we have,

$$a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

Or, changing the order of the terms, and factoring, we have,

$$a^2 + 2a(b+c+d) + b^2 + 2b(c+d) + c^2 + 2cd + d^2. \text{ Hence, the}$$

RULE.—To the sum of the squares of the terms add twice the product of each pair of terms.

Or, To the square of each term add twice its product into the sum of all the terms which follow it.

20. Required the square of  $x + y + z$ .

21. Required the square of  $a - b + c$ .

22. Required the square of  $a + x + y + z$ .

---

273. How may a polynomial be raised to any required power? 274. What is the rule for squaring a polynomial?

**275.** When one of the terms of a binomial is a *fraction*, it may be involved by actual multiplication, or by reducing the mixed quantity to an improper fraction, and then involving the fraction. (Art. 171.)

23. Required the square of  $x + \frac{1}{2}$ ; and  $x - \frac{1}{2}$ .

$$\begin{array}{r} x + \frac{1}{2} \\ x + \frac{1}{2} \\ \hline x^2 + \frac{1}{2}x \\ + \frac{1}{2}x + \frac{1}{4} \\ \hline x^2 + x + \frac{1}{4} \end{array}$$

$$\begin{array}{r} x - \frac{1}{2} \\ x - \frac{1}{2} \\ \hline x^2 - \frac{1}{2}x \\ - \frac{1}{2}x + \frac{1}{4} \\ \hline x^2 - x + \frac{1}{4} \end{array}$$

Or, reduce the mixed quantities to improper fractions. Thus,

$$x + \frac{1}{2} = \frac{2x+1}{2}; \quad \text{and} \quad x - \frac{1}{2} = \frac{2x-1}{2}. \quad (\text{Art. 171.})$$

$$\left(\frac{2x+1}{2}\right)^2 = \frac{4x^2+4x+1}{4}; \quad \text{and} \quad \left(\frac{2x-1}{2}\right)^2 = \frac{4x^2-4x+1}{4}.$$

Expand the following mixed quantities:

24.  $(a + \frac{2}{3})^2$ .

26.  $(-\frac{6}{7} + 2abc)^2$ .

25.  $(a - \frac{c}{2})^2$ .

27.  $(-\frac{b}{m} + 3xy)^2$ .

**276.** Powers are *added and subtracted* like other quantities. (Arts. 67, 77.) For, the *same* powers of the *same* letters are *like quantities*; while powers of *different* letters and *different* powers of the *same* letter are *unlike* quantities, and are treated accordingly. (Arts. 43, 44.)

28. To  $7a^2 + 5(a+b)^3 - 6x + 3x^3 + a^5$   
 Add  $-3a^2 + 4(a+b)^3 + 4x + 4x^3 - a^4$   
 Ans.  $4a^2 + 9(a+b)^3 - 2x + 3x^3 + 4x^2 + a^5 - a^4$

29. From  $3a^3 + 5b^2 - 4c^3 + 4x^2 - a^5$   
 Take  $-4a^3 + 3b^2 + 3c^3 - 5x^3 + a^4$   
 Ans.  $7a^3 + 2b^2 - 7c^3 + 5x^3 + 4x^2 - a^5 - a^4$

275. How involve a binomial, when one term is a fraction? 276. How are powers added and subtracted? Why?

## MULTIPLICATION OF POWERS.

**277. To Multiply Powers of the Same Root.**

1. What is the product of  $3a^4b^3$  multiplied by  $a^2b^2$ ?

SOLUTION.—Adding the exponents of each letter, we have  $3a^6$  and  $b^5$ . Now  $3a^6 \times b^5 = 3a^6b^5$ , *Ans.* (Art. 94.)

2. Multiply  $3a^5b^4$  by  $a^{-3}b^{-2}$ .

SOLUTION.—Adding the exponents of each letter, as before, we have  $3a^2b^2$ , *Ans.* Hence, the

RULE.—*Add the exponents of the given quantities, and the result will be the product.* (Art. 94.)

NOTES.—1. This rule is applicable to *positive* and *negative* exponents.

2. Powers of different roots are multiplied by writing them one after another.

Multiply the following powers:

3.  $a^6$  by  $a^2$ .

7.  $a^{-4}b$  by  $a^{-3}b^4$ .

4.  $x^{-5}$  by  $x^{-3}$ .

8.  $a^{-4}cd$  by  $a^7c^3d^2$ .

5.  $b^{-7}$  by  $b^4$ .

9.  $b^4c^{-6}y^{-2}$  by  $b^{-2}c^3y^4$ .

6.  $a^m$  by  $a^n$ .

10.  $a^6y^{-4}z^3$  by  $a^{-3}y^{-5}z^4$ .

## DIVISION OF POWERS.

**278. To Divide Powers of the Same Root.**

11. Divide  $a^5$  by  $a^3$ .

SOLUTION.—Subtracting one exponent from the other, we have  $a^5 \div a^3 = a^2$ , the quotient sought. (Art. 113.) Hence, the

RULE.—*Subtract the exponent of the divisor from that of the dividend; the result is the quotient.* (Art. 113.)

NOTE.—This rule is applicable to *positive* and *negative* exponents.

---

277. How multiply powers of the same root? *Note.* Of different roots? 278. What is the rule for dividing powers of the same root?

Divide the following powers :

- |                         |   |
|-------------------------|---|
| 12. $a^8$ by $a^{-8}$ . | 16. $x^4yz^2$ by $x^{-3}y^4z^{-5}$ .      |
| 13. $x^{-8}$ by $x^3$ . | 17. $12a^3b^{-2}c$ by $3a^2b^{-3}c^5$ .   |
| 14. $b^4$ by $b^2$ .    | 18. $6x^4y^2z^3$ by $2x^{-2}yz^2$ .       |
| 15. $c^{-5}$ by $c^8$ . | 19. $60a^3b^4c^5$ by $5a^{-2}b^4c^{-8}$ . |

**279.** The *Method* of denoting *Reciprocal Powers* shows that *any factor* may be transferred from the *numerator* of a fraction to the *denominator*, and *vice versa*, by *changing the sign of its exponent* from  $+$  to  $-$ , or  $-$  to  $+$ . (Art. 256.)

20. Transfer the denominator of  $\frac{a^5}{x^3}$  to the numerator.

SOLUTION.  $\frac{a^5}{x^3} = a^5 \times \frac{1}{x^3} = a^5 \times x^{-3} = a^5x^{-3}$ , Ans.

21. Transfer the denominator of  $\frac{a}{x^{-6}}$  to the numerator.

SOLUTION.  $\frac{a}{x^{-6}} = \frac{a}{\frac{1}{x^6}} = a \times \frac{1}{\frac{1}{x^6}} = a \times x^6 = ax^6$ , Ans.

22. Transfer the numerator of  $\frac{a^5}{y}$  to the denominator.

SOLUTION.  $\frac{a^5}{y} = \frac{1}{y} \times a^5 = \frac{1}{y} + \frac{1}{a^5} = \frac{1}{y} + a^{-5} = \frac{1}{a^{-5}y}$ , Ans.

23. Transfer the numerator of  $\frac{a^{-6}}{y}$  to the denominator.

SOLUTION.  $\frac{a^{-6}}{y} = \frac{1}{a^6} \times \frac{1}{y} = \frac{1}{a^6y}$ , Ans.

24. Transfer  $x^{-8}$  to the denominator of  $\frac{ax^{-8}}{y}$ .

25. Transfer  $y^4$  to the numerator of  $\frac{a}{by^4}$ .

26. Transfer  $d^{-5}$  to the denominator of  $\frac{ad^{-5}}{x^2}$ .

27. Transfer  $x^n$  to the numerator of  $\frac{b}{ax^n}$ .

---

279. What inference may be drawn from the method of denoting reciprocal powers? How transfer a factor?

## CHAPTER XIII

### EVOLUTION.\*

**280.** *Evolution* is finding a *root* of a quantity. It is often called the *Extraction* of roots.

**281.** A *Root* is one of the *equal factors* of a quantity.

NOTES.—1. *Powers* and *roots* are correlative terms. If one quantity is a *power* of another, the latter is a *root* of the former.

Thus,  $a^3$  is the cube of  $a$ , and  $a$  is the cube root of  $a^3$ .

2. The learner should observe the following distinctions :

1st. By *involution* a product of *equal factors* is found.

2d. By *evolution* a quantity is resolved into *equal factors*. It is the reverse of involution.

3d. By *division* a quantity is resolved into *two factors*.

4th By *subtraction* a quantity is separated into *two parts*.

**282.** *Roots*, like *powers*, are divided into *degrees* ; as, the square, or second root; the cube, or third root; the fourth root, etc.

**283.** The *Square Root* is one of the *two equal factors* of a quantity.

Thus,  $5 \times 5 = 25$ , and  $a \times a = a^2$ ; therefore 5 is the square root of 25, and  $a$  the square root of  $a^2$ .

**284.** The *Cube Root* is one of the three equal factors of a quantity.

Thus,  $3 \times 3 \times 3 = 27$ , and  $a \times a \times a = a^3$ ; therefore, 3 is the cube root of 27, and  $a$  is the cube root of  $a^3$ .

---

280. What is evolution? 281. A root? *Note.* Of what is evolution the reverse?  
283. What is the square root? 284. Cube root?

\*. From the Latin *evolvere*, to unfold.

**285.** Roots are denoted in two ways:

- 1st. By prefixing the *radical* sign  $\sqrt{\phantom{x}}$  to the quantity.\*
- 2d. By placing a *fractional* exponent on the right of the quantity.

Thus,  $\sqrt[2]{a}$  and  $a^{\frac{1}{2}}$  denote the square root of  $a$ .

$\sqrt[3]{a}$  and  $a^{\frac{1}{3}}$  denote the cube root of  $a$ , etc.

NOTES.—1. The figure placed over the radical sign, is called the *Index of the Root*, because it denotes the *name* of the root.

Thus,  $\sqrt[2]{a}$ , and  $\sqrt[3]{a}$ , denote the square and cube root of  $a$ .

2. In expressing the *square* root, it is customary to use simply the radical sign  $\sqrt{\phantom{x}}$ , the 2 being understood.

Thus, the expression  $\sqrt{25} = 5$ , is read, “the square root of 25 = 5.”

3. The method of expressing roots by *fractional exponents* is derived from the manner of denoting powers by *integral* indices.

Thus,  $a^4 = a \times a \times a \times a$ ; hence, if  $a^4$  is divided into *four equal* factors, *one* of these equal factors may properly be expressed by  $a$ .

**286.** The *numerator* of a fractional exponent denotes the *power*, and the *denominator* the root.

Thus,  $a^{\frac{1}{3}}$  denotes the *cube* root of the *first* power of  $a$ ; and  $a^{\frac{4}{3}}$  denotes the *fourth* root of the *third* power of  $a$ , or the *third* power of the *fourth* root, etc.

Read the following expressions:

- |                        |                        |                        |                         |
|------------------------|------------------------|------------------------|-------------------------|
| 1. $a^{\frac{1}{2}}$ . | 4. $b^{\frac{5}{3}}$ . | 7. $d^{\frac{1}{4}}$ . | 10. $a^{\frac{m}{n}}$ . |
| 2. $a^{\frac{1}{3}}$ . | 5. $c^{\frac{2}{3}}$ . | 8. $m^{\frac{5}{3}}$ . | 11. $a^{\frac{4}{n}}$ . |
| 3. $a^{\frac{1}{5}}$ . | 6. $x^{\frac{1}{2}}$ . | 9. $n^{\frac{7}{3}}$ . | 12. $x^{\frac{m}{n}}$ . |

13. Write the third root of the fourth power of  $a$ .

14. Write the fifth power of the fourth root of  $x$ .

15. Write the eighth root of the twelfth power of  $y$ .

**287.** A *Perfect Power* is one whose exact root can be found. This root is called a *rational quantity*.

285. How are roots denoted? 286. What does the numerator of a fractional exponent denote? The denominator? 287. What is a perfect power?

\* From the Latin *radix*, a root.

The sign  $\sqrt{\phantom{x}}$  is a corruption of the letter  $r$ , the initial of *radix*.

**288.** An *Imperfect Power* is a quantity whose exact root cannot be found.

**289.** A *Surd* is the root of an imperfect power. It is often called an *irrational quantity*.

Thus, 5 is an imperfect power, and its square root,  $2.23+$ , is a surd.

NOTE.—All *roots* as well as *powers* of 1, are 1. For, a root is a factor, which multiplied into itself produces a power; but no number except 1 multiplied into itself can produce 1. (Art. 272.)

Thus, 1,  $1^2$ ,  $1^n$ , and  $\sqrt{1}$ ,  $\sqrt[3]{1}$ ,  $\sqrt[n]{1}$ , etc., are all equal.

**290.** *Negative Exponents* are used in expressing roots as well as powers. (Arts. 255, 257.)

Thus,  $\frac{1}{a^{\frac{1}{2}}} = a^{-\frac{1}{2}}$ ;  $\frac{1}{a^{\frac{1}{4}}} = a^{-\frac{1}{4}}$ ;  $\frac{1}{a^{\frac{1}{n}}} = a^{-\frac{1}{n}}$ .

**291.** The *value* of a quantity is not altered if the *index* of the power or root is exchanged for any other index of the same value.

Thus, instead of  $x^{\frac{1}{2}}$ , we may employ  $x^{\frac{2}{4}}$ , etc. Hence,

**292.** A *fractional* exponent may be expressed in *decimals*.

Thus,  $a^{\frac{1}{2}} = a^{.5} = a^{0.5}$ . That is, the square root of  $a$  is equal to the fifth power of the tenth root of  $a$ .

Express the following exponents in decimals:

- |  |  |
|--|--|
| 16. Write $a^{\frac{1}{2}}$ in decimals.   | 19. Write $b^{\frac{1}{2}}$ in decimals. |
| 17. Write $a^{\frac{1}{3}}$ in decimals.   | 20. Write $x^{\frac{1}{2}}$ in decimals. |
| 18. Write $a^{\frac{2}{3}}$ in decimals.   | 21. Write $y^{\frac{2}{3}}$ in decimals. |
| 22. Express $a^{\frac{1}{2}}$ in decimals. | Ans. $a^{\frac{1}{2}} = a^{0.33333+}$ .  |
| 23. Express $x^{\frac{2}{3}}$ in decimals. | Ans. $x^{\frac{2}{3}} = x^{0.66666+}$ .  |
| 24. Express $y^{\frac{1}{2}}$ in decimals. | Ans. $y^{\frac{1}{2}} = y^{1.25}$ .      |
| 25. Write $a^{\frac{3}{2}}$ in decimals.   | Ans. $a^{\frac{3}{2}} = a^{1.5}$ .       |

NOTE.—In many cases, a *fractional exponent* can only be expressed *approximately* by *decimals*.

---

288. An imperfect power? 289. A surd? 290. Are negative exponents used in expressing roots? 292. How are fractional exponents sometimes expressed?

**293.** The *Signs of Roots* are governed by the following

### PRINCIPLES.

1°. *An odd root of a quantity has the same sign as the quantity.*

2°. *An even root of a positive quantity is either positive or negative, and has the double sign,  $\pm$ .*

Thus, the square of  $+a$  is  $a^2$ , and the square of  $-a$  is  $a^2$ ; therefore the square root of  $a^2$  may be either  $+a$  or  $-a$ ; that is,  $\sqrt{a^2} = \pm a$ .

3°. *The root of the product of several factors is equal to the product of their roots.*

NOTES.—1. The ambiguity of an even root is removed, when it is known whether the power arises from a positive or a negative quantity.

2. It should also be observed that the two square roots of a positive quantity are numerically equal, but have contrary signs.

**294.** An *Even Root* of a negative quantity cannot be found. It is therefore said to be impossible.

Thus, the square root of  $-a^2$  is neither  $+a$  nor  $-a$ . For,  $+a \times +a = +a^2$ ; and  $-a \times -a = +a^2$ . Hence,

**295.** *An even root of a negative quantity is called an Imaginary Quantity.*

Thus,  $\sqrt{-4}$ ,  $\sqrt{-a^2}$ ,  $\sqrt[4]{-a^2}$ , are imaginary quantities.

**296.** To Find the *Root of a Monomial*.

1. What is the square root of  $a^2$ ?

ANALYSIS.—Since  $a^2 = a \times a$ , it follows that one of the equal factors of  $a^2$  is  $a$ ; therefore,  $a$  is its square root. (Art. 283.)

OPERATION.

$$\sqrt{a^2} = a$$

Again, since multiplying the index of a quantity by a number raises the quantity to a corresponding power, it follows that dividing the index by the same number resolves the quantity into a corresponding root. Thus, dividing the index of  $a^2$  by 2, we have  $a^1$  or  $a$ , which is the square root of  $a^2$ .

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293. What principles govern the signs of roots? When is the double sign used? Illustrate this. *Note.* When is the ambiguity removed? 294. What is an even root of a negative quantity? Illustrate. 295. What is it called?

2. What is the square root of  $9a^4b^2$ ?

ANALYSIS.—Since  $9 = 3 \times 3$ , the index of  $a^4 = 2 \times 2$ , and the index of  $b^2 = 1 \times 2$ , it follows that the square root of 9 is 3, that of  $a^4$  is  $a^2$ , and that of  $b^2$  is  $b^1$  or  $b$ . Therefore,  $\sqrt{9a^4b^2} = 3a^2b$ . Hence, the

OPERATION.

$$\sqrt{9a^4b^2} = 3a^2b$$

**RULE.**—Divide the index of each letter by the index of the required root; to the result prefix the root of the coefficient with the proper sign. (Art. 293.)

**NOTE.**—This rule is based upon Principle 3. If a quantity is an imperfect power, its root can only be indicated.

296. *a.* The root of a Fraction is found by extracting the root of each of its terms.

Find the required roots of the following quantities:

3.  $\sqrt{a^6}$ .

10.  $\sqrt{36a^4b^2}$ .

4.  $\sqrt[4]{a^1}$  or  $a$ .

11.  $\sqrt[3]{2x^2y^3}$ .

5.  $\sqrt[3]{4xy}$ .

12.  $\sqrt{64a^8b^6}$ .

6.  $\sqrt[3]{8a^5b^6}$ .

13.  $\sqrt[5]{13xy}$ .

7.  $\sqrt[3]{27abc}$ .

14.  $\sqrt{49x^4y^2}$ .

8.  $\sqrt[4]{16a^4n}$ .

15.  $\sqrt[3]{27a^3b^3}$ .

9.  $\sqrt[5]{3a^3x^{10}}$ .

16.  $\sqrt{\frac{49x^4}{64y^3}}$ .

## 297. To Extract the Square Root of the Square of a Binomial.

1. Required the square root of  $a^2 + 2ab + b^2$ .

ANALYSIS.—Arrange the terms according to the powers of the letter  $a$ ; the square root of the first term is  $a$ , which is the first term of the root. Next, subtracting its square from the given quantity, bring down the remainder,  $2ab + b^2$ .

OPERATION.

$$\begin{array}{r} a^2 + 2ab + b^2 \quad (a + b \\ \underline{a^2} \phantom{+ 2ab + b^2} \\ 2a + b) \quad 2ab + b^2 \\ \underline{2ab + b^2} \\ 0 \end{array}$$

296. How find the root of a monomial? *Note.*—Upon what principle is this rule based? 296. *a.* How find the root of a fraction?

Divide the 1st term of this rem. by  $2a$ , double the root thus found, the quotient  $b$  is the other term of the root. Place  $b$  both in the root and on the right of the divisor. Finally, multiply the divisor, thus increased, by the second term of the root, and subtracting the product from  $2ab + b^2$ , there is no remainder. Therefore,  $a + b$  is the root required.

The square root of  $a^2 - 2ab + b^2$  is found in the same manner, the terms of the root being connected by the sign  $-$ . Hence, the

**RULE.**—*Find the square roots of the first and third terms, and connect them by the sign of the middle term.*

2. What is the square root of  $x^2 + 4x + 4$ ?
3. What is the square root of  $a^2 - 2a + 1$ ?
4. What is the square root of  $1 + 2x + x^2$ ?
5. What is the square root of  $x^2 + \frac{4}{3}x + \frac{4}{9}$ ?
6. What is the square root of  $a^2 - a + \frac{1}{4}$ ?
7. What is the square root of  $x^2 + bx + \frac{b^2}{4}$ ?

## 298. To Extract the Square Root of a Polynomial.

8. Required the square root of  $4a^4 - 12a^3 + 5a^2 + 6a + 1$ .

OPERATION.

$$\begin{array}{r}
 4a^4 - 12a^3 + 5a^2 + 6a + 1 \quad (2a^2 - 3a - 1 \\
 \underline{4a^4} \\
 4a^2 - 3a \quad ) - 12a^3 + 5a^2 + 6a + 1 \\
 \quad \underline{- 12a^3 + 9a^2} \\
 4a^2 - 8a - 1 \quad ) - 4a^2 + 6a + 1 \\
 \quad \underline{- 4a^2 + 6a + 1}
 \end{array}$$

**ANALYSIS.**—The square root of the first term is  $2a^2$ , which is the first term of the root. Subtract its square from the term used and bring down the remaining terms. Divide the remainder by double the root thus found; the quotient  $-3a$  is the next term of the root, and is placed both in the root and on the right of the partial divisor. Multiply the divisor thus increased by the term last placed in the root, and subtract the product as before.

Next, divide the remainder by twice the part of the root already found, and the quotient is  $-1$ , which is placed both in the root and on the right of the divisor.

Finally, multiply the divisor, thus increased, by the term last placed in the root, and subtracting the product, as before, there is no remainder. Therefore, the required root is  $2a^2 - 3a - 1$ . Hence, the

† **RULE.—I.** *Arrange the terms according to the powers of some letter, beginning with the highest, find the square root of the first term for the first term of the root, and subtract its square from the given quantity.*

*II. Divide the first term of the remainder by double the root already found, and place the quotient both in the root and on the right of the divisor.*

*III. Multiply the divisor thus increased by the term last placed in the root, and subtract the product from the last dividend. If there is a remainder, proceed with it as before, till the root of all the quantities is found.*

**PROOF.**—*Multiply the root by itself, as in arithmetic.*

**NOTE.**—This rule is essentially the same as that used for extracting the square root of numbers.

Extract the square root of the following quantities:

9.  $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$ .
10.  $a^2 - 4ab + 2a + 4b^2 - 4b + 1$ .
11.  $a^4 + 4a^2b + 4b^2 - 4a^2 - 8b + 4$ .
12.  $1 - 4b^2 + 4b^4 + 2x - 4b^2x + x^2$ .
13.  $4a^4 - 16a^3 + 24a^2 - 16a + 4$ .
14.  $a^2 - ab + \frac{1}{4}b^2$ .
15.  $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2}$ .

**299.** The *fourth* root of a quantity may be found by extracting the square root *twice*; that is, by extracting the square root of the square root.

Thus,  $\sqrt{16a^4} = 4a^2$ , and  $\sqrt{4a^2} = 2a$ . Therefore,  $2a$  is the fourth root of  $16a^4$ .

**PROOF.**  $2a \times 2a = 4a^2$ ;  $4a^2 \times 4a^2 = 16a^4$ .

The *eighth*, the *sixteenth*, etc., roots may be found in like manner.

## CHAPTER XIV.

### RADICAL QUANTITIES.

**300.** A *Radical* is the root of a quantity indicated by the radical sign or fractional exponent.

NOTES.—1. The *figures* or *letters* placed before radicals are *coefficients*.

2. In the following investigations, all quantities placed under the radical sign, or having a fractional exponent, whether *perfect* or *imperfect* powers, are treated as *radicals*, unless otherwise mentioned.

**301.** The *Degree* of a radical is denoted by its *index*, or by the *denominator* of its fractional exponent. (Arts. 285, 286.)

Thus,  $\sqrt[3]{ax}$ ,  $a^{\frac{1}{2}}$ , and  $(a+b)^{\frac{1}{3}}$ , are radicals of the same degree.

**302.** *Like Radicals* are those which express the *same* root of the *same* quantity. Hence, *like* radicals are *like* quantities. (Art. 43.)

Thus,  $5\sqrt{a^2-b}$  and  $3\sqrt{a^2-b}$ , etc., are like radicals.

### REDUCTION OF RADICALS.

**303.** *Reduction of Radicals* is changing their form without altering their value.

**304.** The *Simplest Form* of radicals is that which contains no *factor* whose indicated root can be extracted. Hence, in reducing them to their simplest form, *all exact powers* of the same name as the root must be removed from under the radical sign.

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300. What is a radical? 301. How is the degree of a radical denoted? 302. What are like radicals? 303. Define reduction of radicals. 304. What is the simplest form of radicals?

## CASE I.

## 305. To Reduce a Radical to its Simplest Form.

1. Reduce  $\sqrt{18a^2x}$  to its simplest form.

ANALYSIS.—By inspection, we perceive that the given radical is composed of two factors,  $9a^2$  and  $2x$ , the first being a perfect square and the second a surd. (Art. 289.) Removing  $9a^2$  from under the radical sign and extracting its square root, we have  $3a$ , which prefixed to the other factor gives  $3a\sqrt{2x}$ , the simplest form required.

OPERATION.

$$\begin{aligned}\sqrt{18a^2x} &= \sqrt{9a^2 \times 2x} \\ &= \sqrt{9a^2} \times \sqrt{2x} \\ \therefore \sqrt{18a^2x} &= 3a\sqrt{2x}\end{aligned}$$

2. Reduce  $4\sqrt[3]{a^4 - a^3x}$  to its simplest form.

ANALYSIS. — Factoring the radical part, we have the two factors,  $\sqrt[3]{a^3}$  and  $\sqrt[3]{a - x}$ , the first being a perfect cube, and the second a surd. Remove  $a^3$  from under the radical sign, and its root is  $a$ , which multiplied by the coefficient 4, and prefixed to the radical part, gives  $4a\sqrt[3]{a - x}$ , the simplest form. Hence, the

OPERATION.

$$\begin{aligned}4\sqrt[3]{a^4 - a^3x} &= 4\sqrt[3]{a^3 \times (a - x)} \\ &= 4\sqrt[3]{a^3} \times \sqrt[3]{a - x} \\ \therefore 4\sqrt[3]{a^4 - a^3x} &= 4a\sqrt[3]{a - x}\end{aligned}$$

RULE.—I. *Resolve the radical into two factors, one of which is the greatest power of the same name as the root.*

II. *Extract the root of this power, and multiplying it by the coefficient, prefix the result to the other factor, with the radical sign between them.*

NOTES.—I. This rule is based upon the principle that the root of the product of two or more factors is equal to the product of their roots.

2. When the radicals are small, the greatest exact power they contain may be readily found by inspection.

3. Reduce  $3\sqrt{50a^2x}$  to its simplest form. *Ans.*  $15a\sqrt{2x}$ .

4. Reduce  $6\sqrt[3]{54x^3y}$  to its simplest form. *Ans.*  $18x\sqrt[3]{2y}$ .

### 306. To Find in large Radicals the Greatest Power corresponding to the indicated Root.

5. Reduce  $\sqrt{1872}$  to its simplest form.

OPERATION.

$$4 \overline{) 1872}$$

$$4 \overline{) 468}$$

$$9 \overline{) 117} ( 13$$

$$\sqrt{1872} = \sqrt{4 \times 4 \times 9} \times \sqrt{13}$$

$$" = \sqrt{144} \times \sqrt{13}$$

$$\sqrt{1872} = 12\sqrt{13}, \text{ Ans.}$$

ANALYSIS.—Divide the radical by the smallest power of the same degree that is a factor of it; the quotient is 468. Divide this quantity by 4; the second quotient is 117. The smallest power of the same degree that will divide 117, is 9. The quotient is 13, which is not divisible by any power of the same degree. The product of the divisors,  $4 \times 4 \times 9 = 144$ , is the greatest square of the given radical. Extracting the square root of 144, we have  $12\sqrt{13}$ , the simplest form required. Hence, the

RULE.—Divide the radical by the smallest power of the same degree which is a factor of the given radical.

Divide this quotient as before; and thus proceed till a quotient is obtained which is not divisible by any power of the same degree. The continued product of the divisors will be the greatest power required.

NOTE.—This rule is founded on the principle that the product of any two or more square numbers is a square, the product of any two or more cubic numbers is a cube, etc.

Thus,  $2^2 \times 3^2 = 36 = 6^2$ ; and  $2^3 \times 3^3 = 216 = 6^3$ .

Reduce the following radicals to their simplest form:

$$6. \sqrt{a^2b}.$$

$$7. \sqrt{8a^2b}.$$

$$8. 2\sqrt{9xy}.$$

$$9. 3\sqrt[3]{24}.$$

$$10. 5\sqrt[3]{135}.$$

$$11. 6\sqrt{252a^2b}.$$

$$12. \sqrt[3]{54a^3c}.$$

$$13. 7\sqrt{9a^2 - 27a^2b}.$$

$$14. \sqrt[3]{64x^3y}.$$

$$15. \sqrt[4]{81a^4b}.$$

$$16. \sqrt{468a^2c}.$$

$$17. \sqrt{1584a^2}.$$

306. How find the greatest power corresponding to the indicated root, in large radicals? Note. On what principle is this rule founded?

## CASE II.

## 307. To Reduce a Rational Quantity to the Form of a Radical.

1. Reduce  $3a^2$  to the form of the cube root.

ANALYSIS.—The cube root of a quantity, we have seen, is one of its three equal factors. (Art. 284.) Now  $3a^2$  raised to the third power is  $27a^6$ . Therefore  $3a^2 = \sqrt[3]{27a^6}$ . Hence, the

OPERATION.

$$(3a^2)^3 = 27a^6$$

$$\therefore 3a^2 = \sqrt[3]{27a^6}$$

RULE.—Raise the quantity to the power denoted by the given root, and to the result prefix the corresponding radical sign.

NOTE.—The coefficient of a radical, or any factor of it, may be placed under the sign, by raising it to the corresponding power, and placing it as a factor under the radical sign.

2. Reduce  $2a^3b$  to the form of the cube root.
3. Reduce  $(2a + b)$  to the form of the square root.
4. Reduce  $(a - 2b)$  to the form of the square root.
5. Place the coefficient of  $3a\sqrt{b}$  under the radical sign.
6. Place the coefficient of  $2a^2\sqrt[3]{ab}$  under the radical sign.
7. Reduce  $2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$  to the form of the fourth root.
8. Reduce  $\frac{1}{3}abc$  to the form of the cube root.
9. Reduce  $3(a - b)$  to the form of the cube root.
10. Reduce  $a^2$  to the form of the cube root.

NOTE.—When a power is to be raised to the form of a required root, it is not the given letter that is to be raised, but the power of the letter.

11. Reduce  $a^3c^3$  to the form of the fourth root.
12. Reduce  $a - b$  to the form of the square root.
13. Reduce  $a^m$  to the form of the  $n$ th root.

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307. How reduce a rational quantity to the form of a radical? *Note.* How place a coefficient under the radical sign? *Note.* How raise a power to the form of a required root?

## CASE III.

**308. To Reduce Radicals of different Degrees to others of equal Value, having a Common Index.**

I. Reduce  $a^{\frac{1}{2}}$  and  $b^{\frac{1}{3}}$  to equivalent radicals of a common index.

ANALYSIS.—The fractional indices  $\frac{1}{2}$  and  $\frac{1}{3}$ , reduced to a common denominator become  $\frac{2}{6}$  and  $\frac{1}{6}$ . But  $a^{\frac{2}{6}} = (a^{\frac{1}{3}})^{\frac{2}{2}}$ , and  $b^{\frac{1}{6}} = (b^{\frac{1}{3}})^{\frac{1}{2}}$ . (Art. 174.)

OPERATION.

$$\begin{aligned} \frac{1}{2} &= \frac{2}{4} \text{ and } \frac{1}{3} = \frac{1}{3} \\ \therefore a^{\frac{1}{2}} &= a^{\frac{2}{4}} \text{ and } b^{\frac{1}{3}} = b^{\frac{1}{3}} \\ a^{\frac{2}{4}} &= (a^{\frac{1}{2}})^{\frac{2}{2}} \text{ and } b^{\frac{1}{3}} = (b^{\frac{1}{3}})^{\frac{1}{1}} \end{aligned}$$

Therefore  $(a^{\frac{1}{2}})^{\frac{2}{2}}$  and  $(b^{\frac{1}{3}})^{\frac{1}{1}}$  are the radicals required. Hence, the

RULE.—I. Reduce the indices to a common denominator.

II. Raise each quantity to the power expressed by the numerator of the new index, and indicate the root expressed by the common denominator. (Art. 174.)

Reduce the following radicals to a common index:

2.  $a^{\frac{1}{2}}$  and  $(bc)^{\frac{1}{3}}$ .

7.  $\sqrt{4a^3}$  and  $\sqrt[3]{2a^2}$ .

3.  $3^{\frac{1}{2}}$  and  $5^{\frac{1}{3}}$ .

8.  $a^{\frac{1}{n}}$  and  $b^{\frac{1}{m}}$ .

4.  $a^{\frac{1}{2}}$  and  $6^{\frac{1}{3}}$ .

9.  $b^{\frac{1}{2}}$  and  $c^{\frac{1}{3}}$ .

5.  $\sqrt{5}$ ,  $\sqrt[3]{3}$ , and  $\sqrt[4]{2}$ .

10.  $(a+b)^{\frac{1}{2}}$  and  $(a-b)^{\frac{1}{3}}$ .

6.  $\sqrt[3]{2x^4}$  and  $\sqrt{5x^3}$ .

11.  $(x-y)^{\frac{2}{3}}$  and  $(x+y)^{\frac{1}{4}}$ .

## CASE IV.

**309. To Reduce a Quantity to any Required Index.**

I. Reduce  $a^{\frac{1}{2}}$  to the index  $\frac{1}{3}$ .

ANALYSIS.—Divide the index  $\frac{1}{2}$  by  $\frac{1}{3}$ ; we have  $\frac{3}{2}$  or  $1\frac{1}{2}$ . Place this index over  $a$ ; it becomes  $a^{\frac{3}{2}}$ , and setting the required index over this, the result,  $(a^{\frac{3}{2}})^{\frac{1}{3}}$ , is the answer. Hence, the

OPERATION.

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$$

$$\frac{1}{2} \div \frac{1}{3} = \frac{3}{2} = \frac{3}{2}$$

$$\therefore (a^{\frac{3}{2}})^{\frac{1}{3}}, \text{ Ans.}$$

**RULE.**—Divide the index of the given quantity by the required index, and placing the quotient over the quantity, set the required index over the whole.

**NOTE.**—This operation is the same as resolving the original index into two factors, one of which is the required index. (Art. 126, note.)

2. Reduce  $a^{\frac{1}{2}}$  and  $b^{\frac{2}{3}}$  to the index  $\frac{1}{6}$ .

**SOLUTION.**  $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$ , the first index.

$\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3}$ , the second index.

Therefore,  $(a^{\frac{1}{2}})^{\frac{3}{2}}$  and  $(b^{\frac{2}{3}})^{\frac{3}{2}}$  are the quantities required.

3. Reduce  $3^{\frac{1}{2}}$  and  $4^{\frac{1}{3}}$  to the common index  $\frac{1}{6}$ .

4. Reduce  $a^2$  and  $b^3$  to the common index  $\frac{1}{6}$ .

5. Reduce  $a^3$  and  $b^{\frac{1}{2}}$  to the common index  $\frac{3}{2}$ .

6. Reduce  $a^{\frac{1}{2}}$  and  $b^{\frac{2}{3}}$  to the common index  $\frac{1}{6}$ .

7. Reduce  $a^{\frac{1}{m}}$  and  $b^{\frac{2}{n}}$  to the common index  $\frac{1}{n}$ .

## ADDITION OF RADICALS.

### 310. To Find the Sum of two or more Radicals.

1. What is the sum of  $3\sqrt{a}$  and  $5\sqrt{a}$ ?

**ANALYSIS.**—Since these radicals are of the same degree and have the same radical part, they are like quantities.

(Art. 43.) Therefore their coefficients may be added in the same manner as rational quantities. (Art. 67.)

**OPERATION.**

$$3\sqrt{a} + 5\sqrt{a} = 8\sqrt{a}$$

2. What is the sum of  $3\sqrt{8}$  and  $4\sqrt{18}$ ?

**ANALYSIS.**—These radicals are of the same degree, but the radical parts are unlike; therefore, they cannot be united in their present form.

Reducing them to their sim-

plest form, we have  $3\sqrt{8} = 6\sqrt{2}$ , and  $4\sqrt{18} = 12\sqrt{2}$ , which are like radicals. (Arts. 302, 305.) Now  $6\sqrt{2}$  and  $12\sqrt{2} = 18\sqrt{2}$ , Ans.

**OPERATION.**

$$3\sqrt{8} = 3\sqrt{4} \times \sqrt{2} = 6\sqrt{2}$$

$$4\sqrt{18} = 4\sqrt{9} \times \sqrt{2} = 12\sqrt{2}$$

$$\therefore 6\sqrt{2} + 12\sqrt{2} = 18\sqrt{2}$$

309. How reduce quantities to any required index? *Note.* To what is this operation similar?

3. What is the sum of  $3\sqrt{18}$  and  $4\sqrt[3]{24}$ .

ANALYSIS. — Reducing the radicals to their simplest form, we have  $3\sqrt{18} = 9\sqrt{2}$ , and  $4\sqrt[3]{24} = 8\sqrt[3]{3}$ , which are unlike quantities, and can only be added by writing them one after the other, with their proper signs. (Arts. 43, 67.) Hence, the

OPERATION.

$$3\sqrt{18} = 3\sqrt{9} \times \sqrt{2} = 9\sqrt{2}$$

$$4\sqrt[3]{24} = 4\sqrt[3]{8} \times \sqrt[3]{3} = 8\sqrt[3]{3}$$

$$\text{Ans. } 9\sqrt{2} + 8\sqrt[3]{3}$$

RULE.—I. *Reduce the radicals to their simplest form.*

II. *If the radical parts are alike, add the coefficients, and to the sum annex the common radical.*

*If the radicals are unlike, write them one after another, with their proper signs.*

NOTE.—To determine whether radicals are *alike*, it is generally necessary to reduce them to their simplest form. (Art. 305.)

Find the sum of the following radicals:

4.  $\sqrt{12}$  and  $\sqrt{27}$ .

9.  $3\sqrt[3]{54}$  and  $4\sqrt[3]{128}$ .

5.  $\sqrt{20}$  and  $\sqrt{48}$ .

10.  $7\sqrt{243}$  and  $5\sqrt{363}$ .

6.  $2\sqrt{b^3}$  and  $3\sqrt{a^2b}$ .

11.  $a\sqrt{81b}$  and  $3a\sqrt{49b}$ .

7.  $a\sqrt{3a^3b}$  and  $c\sqrt{27ab}$ .

12.  $b\sqrt{25x^2c}$  and  $\sqrt{36x^4c}$ .

8.  $3\sqrt{18ax^2}$  and  $2\sqrt{32a^3}$ .

13.  $4\sqrt[3]{x^6y}$  and  $5\sqrt{x^3y}$ .

## SUBTRACTION OF RADICALS.

**311.** To Find the *Difference* between two Radicals.

I. From  $3\sqrt{45}$  subtract  $2\sqrt{20}$ .

ANALYSIS.—Reducing to the simplest form, we have  $9\sqrt{5}$  and  $4\sqrt{5}$ , which are like quantities. Now  $9\sqrt{5} - 4\sqrt{5} = 5\sqrt{5}$ , the difference required. Hence, the

OPERATION.

$$3\sqrt{45} = 3\sqrt{9} \times \sqrt{5} = 9\sqrt{5}$$

$$2\sqrt{20} = 2\sqrt{4} \times \sqrt{5} = 4\sqrt{5}$$

$$\therefore 3\sqrt{45} - 2\sqrt{20} = 5\sqrt{5}$$

**RULE.**—Reduce the radicals to their simplest form ; change the sign of the subtrahend, and proceed as in addition of radicals. (Art. 310.)

	(2.)	(3.)	(4.)
* From	$4\sqrt{112}$	$\sqrt{480}$	$4\sqrt{320}$
Take	$\sqrt{448}$	$4\sqrt{63}$	$-5\sqrt{80}$

5. From  $3\sqrt{49ax^3}$  take  $2\sqrt{25ax}$ .
6. From  $5\sqrt[3]{a+b}$  take  $3\sqrt[3]{a+b}$ .
7. From  $3\sqrt[3]{b}$  take  $-4\sqrt[3]{b}$ .
8. From  $3\sqrt[3]{250b^4x}$  take  $2\sqrt[3]{54b^4x}$ .
9. From  $-a^{-\frac{1}{2}}$  take  $-2a^{-\frac{1}{2}}$ .
10. From  $5\sqrt{\frac{1}{2}}$  take  $2\sqrt{\frac{1}{2}}$ .

## MULTIPLICATION OF RADICALS.

### 312. To Multiply Radical Quantities.

1. What is the product of  $3\sqrt{a}$  by  $2\sqrt{b}$ .

**ANALYSIS.**—Since these radicals are of the same degree, we multiply the radical parts together, like rational quantities, and to the result prefix the product of the coefficients.

**OPERATION.**

$$3\sqrt{a} \times 2\sqrt{b} = 6\sqrt{ab}$$

2. Multiply  $3\sqrt{a}$  by  $2\sqrt[3]{c}$ .

**ANALYSIS.**—As these radicals are of different degrees, they cannot be multiplied together in their present form. We therefore reduce them to a common index, and then, multiplying as before, we have  $6\sqrt[6]{a^3c^2}$ .

**OPERATION.**

$$\begin{aligned} 3\sqrt{a} &= 3(a^{\frac{1}{2}}) \\ 2\sqrt[3]{c} &= 2(c^{\frac{1}{3}}) \\ \text{Ans. } 6 &(a^{\frac{1}{2}}c^{\frac{1}{3}})^{\frac{1}{6}} \end{aligned}$$

3. Multiply  $a^{\frac{1}{2}}$  by  $a^{\frac{1}{3}}$ .

**ANALYSIS.**—These radicals are of different degrees, but of the same radical part or root ; we therefore multiply them by adding their fractional exponents.  $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ . Therefore,  $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}}$ . Hence, the

**RULE.—I.** Reduce the radicals to a common index.

**II.**—Multiply the radical parts together as rational quantities, and placing the result under the common index, prefix to it the product of the coefficients.

**NOTES.—I.** Roots of like quantities are multiplied together by adding their fractional exponents. (Art. 94.)

2. This rule is based upon the principle that the product of the roots of two or more quantities is the same as the root of the product. (Art. 293, Prin. 3.)

3. The product of radicals becomes rational, whenever the numerator of the index can be divided by its denominator without a remainder.

4. If rational quantities are connected with radicals by the signs + or —, each term in the multiplicand must be multiplied by each term in the multiplier. (Art. 98.)

Multiply the following radicals:

4.  $5\sqrt{18}$  by  $3\sqrt{20}$ .

10.  $a^{\frac{1}{2}}$  by  $x^{\frac{1}{2}}$ .

5.  $a\sqrt{x}$  by  $b\sqrt{x}$ .

11.  $7\sqrt[3]{4}$  by  $3\sqrt[3]{4}$ .

6.  $\sqrt{a+b}$  by  $\sqrt{a-b}$ .

12.  $\sqrt{9a}$  by  $\sqrt{16a}$ .

7.  $\sqrt{ax}$  by  $\sqrt{cy}$ .

13.  $\sqrt{18}$  by  $\sqrt{2}$ .

8.  $a^{\frac{2}{3}}$  by  $c^{\frac{1}{3}}$ .

14.  $\sqrt{8ax}$  by  $\sqrt{2ax}$ .

(9.)

Multiply  $a + \sqrt{b}$

By  $c + \sqrt{d}$

$$\begin{array}{r} ac + c\sqrt{b} \\ + a\sqrt{d} + \sqrt{bd} \end{array}$$

Ans.  $ac + c\sqrt{b} + a\sqrt{d} + \sqrt{bd}$

(15.)

Mult.  $a + \sqrt{x}$

By  $1 + b\sqrt{x}$

$$\begin{array}{r} a + \sqrt{x} \\ + ab\sqrt{x} + bx \end{array}$$

Ans.  $a + \sqrt{x} + ab\sqrt{x} + bx$

16.  $2\sqrt{\frac{2}{3}}$  by  $2\sqrt{\frac{1}{2}}$ .

18.  $(m+n)^{\frac{2}{3}}$  by  $(m+n)^{\frac{1}{3}}$

17.  $4\sqrt{\frac{2}{3}}$  by  $3\sqrt{\frac{1}{2}}$ .

19.  $\sqrt{\frac{9ad}{2b}}$  by  $\sqrt{\frac{2ab}{3c}}$ .

312. How multiply radicals? *Notes.* How are roots of like quantities multiplied? Upon what principle is this rule based? When does the product of radicals become rational? If radicals are connected with rational quantities, how multiply them?

## DIVISION OF RADICALS.

313. To *Divide* Radical Quantities.1. Divide  $4\sqrt{24ac}$  by  $2\sqrt{8a}$ .

ANALYSIS.—Since the given radicals are of the same degree, one may be divided by the other, like rational quantities, the quotient being  $\sqrt{3c}$ . (Art. 111.) To this result prefixing the quotient of one coefficient divided by the other, we have  $2\sqrt{3c}$ , the quotient required.

OPERATION.

$$\frac{4\sqrt{24ac}}{2\sqrt{8a}} = 2\sqrt{3c}$$

2. Divide  $4\sqrt{ac}$  by  $2\sqrt[3]{a}$ .

ANALYSIS.—Since these radicals are of different degrees, they cannot be divided in their present form. We therefore reduce them to a common index, then divide one by the other, and to the result prefix the quotient of the coefficients. The answer is  $2\sqrt[6]{ac^3}$ .

$$\begin{aligned} \frac{4\sqrt{ac}}{2\sqrt[3]{a}} &= \frac{4(ac)^{\frac{1}{2}}}{2(a)^{\frac{1}{3}}} \\ \therefore \frac{4\sqrt{ac}}{2\sqrt[3]{a}} &= \frac{4(a^3c^3)^{\frac{1}{6}}}{2(a^2)^{\frac{1}{6}}} \\ &\text{or } 2(ac^3)^{\frac{1}{6}}, \text{ Ans.} \end{aligned}$$

3. Divide  $a^{\frac{1}{2}}$  by  $a^{\frac{1}{3}}$ .

ANALYSIS.—These radicals are of different degrees, but have the same radical part or root. We therefore divide them by subtracting the fractional exponent of the divisor from that of the dividend. (Art. 113.) Reducing the exponents to a common denominator,  $a^{\frac{1}{2}} = a^{\frac{3}{6}}$ , and  $a^{\frac{1}{3}} = a^{\frac{2}{6}}$ . Now  $a^{\frac{3}{6}} \div a^{\frac{2}{6}} = a^{\frac{1}{6}}$ , Ans. Hence, the

OPERATION.

$$\begin{aligned} a^{\frac{1}{2}} &= a^{\frac{3}{6}} \\ a^{\frac{1}{3}} &= a^{\frac{2}{6}} \\ a^{\frac{3}{6}} \div a^{\frac{2}{6}} &= a^{\frac{1}{6}} \\ \therefore a^{\frac{1}{2}} \div a^{\frac{1}{3}} &= a^{\frac{1}{6}} \end{aligned}$$

RULE.—I. *Reduce the radical parts to a common index.*

II. *Divide one radical part by the other, and placing the quotient under the common index, prefix to the result the quotient of their coefficients.*

NOTE.—Roots of like quantities are divided by subtracting the fractional exponent of the divisor from that of the dividend. (Art. 113.)

Divide the following radicals:

- |   |  |
|---|--|
| 4. $\sqrt{12a^3c}$ by $\sqrt{4c}$ .                     | 10. $14a\sqrt{xy}$ by $7\sqrt{y}$ .                  |
| 5. $6\sqrt{bdx^3}$ by $2\sqrt{dx}$ .                    | 11. $(a+b)^{\frac{3}{2}}$ by $(a+b)^{\frac{1}{2}}$ . |
| 6. $(a^3+ax)^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$ .      | 12. $3\sqrt{50x^4}$ by $\sqrt{2x}$ .                 |
| 7. $12(a^2y^2)^{\frac{1}{4}}$ by $(ay)^{\frac{1}{4}}$ . | 13. $\sqrt[3]{x^3-y^3}$ by $\sqrt[3]{x+y}$ .         |
| 8. $24b\sqrt{ax}$ by $8\sqrt{a}$ .                      | 14. $16\sqrt{32}$ by $2\sqrt{4}$ .                   |
| 9. $15ac\sqrt{bx}$ by $2c\sqrt{x}$ .                    | 15. $8\sqrt{512}$ by $4\sqrt{2}$ .                   |

## INVOLUTION OF RADICALS.

### 314. To Involve a Radical to any required Power.

1. Find the square of  $a^{\frac{1}{2}}$ .

ANALYSIS.—As a square is the product of two equal factors, we multiply the given index by the index of the required power. Hence, the

OPERATION.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1$$

$\therefore a^1$ , Ans.

RULE.—*Multiply the index of the root by the index of the required power, and to the result prefix the required power of the coefficient.*

NOTE.—A root is raised to a power of the same name by removing the radical sign or fractional exponent. (Ex. 2.)

- |   |              |
|---|--------------|
| 2. Find the cube of $\sqrt[3]{a+b}$ .               | Ans. $a+b$ . |
| 3. Find the cube of $a^{\frac{1}{2}}$ .             |              |
| 4. What is the square of $3\sqrt{2x}$ .             |              |
| 5. What is the cube of $2\sqrt[3]{a}$ .             |              |
| 6. Required the cube of $\frac{x}{2}\sqrt{2x}$ .    |              |
| 7. Required the cube of $4\sqrt{\frac{ax^3}{4}}$ .  |              |
| 8. Find the fourth power of $3\sqrt{\frac{b}{3}}$ . |              |
| 9. What is the square of $a + \sqrt{y}$ ?           |              |

## EVOLUTION OF RADICALS.

## 315. To Extract the Root of a Radical.

1. Find the cube root of
- $a^3\sqrt[4]{b^3}$
- .

ANALYSIS.—Finding the root of a radical is the same in principle as finding the root of a rational quantity. (Art. 296.) Reducing the index of the radical to an equivalent fractional exponent, we extract the cube root by dividing it by 3. The result is  $ab^{\frac{1}{4}}$ . Hence, the

OPERATION.

$$\sqrt[3]{a^3\sqrt[4]{b^3}} = \sqrt[3]{a^3b^{\frac{3}{4}}}$$

$$\sqrt[3]{a^3b^{\frac{3}{4}}} = ab^{\frac{1}{4}}, \text{ Ans.}$$

RULE.—*Divide the fractional exponent of the radical by the number denoting the required root, and to the result prefix the root of the coefficient.*

NOTES.—1. Multiplying the *index* of a radical by any number is the same as dividing the *fractional exponent* by that number.

Thus,  $\sqrt[3]{a} = a^{\frac{1}{3}}$ . Multiplying the former by 2, and dividing the latter by 2, we have  $\sqrt[6]{a} = a^{\frac{1}{6}}$ .

2. If the coefficient is not a perfect power, it should be placed under the radical sign and be reduced to its simplest form. (Art. 305.)

2. Required the square root of  $9\sqrt[3]{a^3}$ .
3. Required the square root of  $4\sqrt[3]{3x}$ .
4. Find the cube root of  $3\sqrt{xy}$ .
5. Find the cube root of  $2b\sqrt{2b}$ .
6. What is the cube root of  $a(bc)^{\frac{1}{2}}$ ?
7. What is the fourth root of  $\frac{2}{3}\sqrt[3]{\frac{2}{3}}$ ?
8. What is the fourth root of  $\sqrt{a^8}\sqrt[3]{c^3}$ ?
9. Find the seventh root of  $128\sqrt[3]{a}$ .
10. Find the fourth root of  $\sqrt{a}(b^{\frac{1}{2}})$ .
11. Find the fifth root of  $4a^2\sqrt{2a}$ .
12. Find the  $n$ th root of  $a\sqrt[5]{bc}$ .

---

315. How extract the root of a radical? *Notes.* To what is multiplying the index of a radical equivalent? If the coefficient is not a perfect power, what is done?

## CHANGING A RADICAL TO A RATIONAL QUANTITY.

## CASE I.

## 316. To Change a Radical Monomial to a Rational Quantity.

1. Change  $\sqrt{a}$  to a rational quantity.

ANALYSIS.—Since multiplying a root of a quantity into itself produces the quantity, it follows that  $\sqrt{a} \times \sqrt{a} = a$ , which is a rational quantity. (Art. 287.)

OPERATION.

$$\sqrt{a} \times \sqrt{a} = a$$

2. It is required to rationalize  $a^{\frac{1}{2}}$ .

ANALYSIS.—A root is multiplied by another root of the same quantity by adding the exponents; therefore we add to the index  $\frac{1}{2}$  such a fraction as will make it equal to 1. (Art. 94.)

OPERATION.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$$

Thus,  $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a$ , the rational quantity required.

3. It is required to rationalize  $x^{\frac{1}{2}}$ .

SOLUTION.—Multiplying  $x^{\frac{1}{2}}$  by  $x^{\frac{1}{2}}$ , the result is  $x$ , which is a rational quantity. Hence, the

OPERATION.

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x$$

RULE.—Multiply the radical by the same quantity having such a fractional exponent as, when added to the given exponent, the sum shall be equal to a unit, or 1.

4. Required a factor which will rationalize  $a^{\frac{1}{2}}$ .
5. What factor will rationalize  $\sqrt[3]{a^2c}$ ?
6. What factor will rationalize  $\sqrt[3]{(a+b)^2}$ ?
7. What factor will rationalize  $\sqrt[3]{a^2b^2c}$ ?
8. What factor will rationalize  $\sqrt[3]{(x+y)^2}$ ?
9. What factor will rationalize  $\sqrt[4]{(a+b)^3}$ ?
10. What factor will rationalize  $\sqrt{(a+b+c)^2}$ ?

## CASE II.

**317. To Change a Radical Binomial to a Rational Quantity.**

1. It is required to rationalize  $\sqrt{a} + \sqrt{b}$ .

ANALYSIS.—The product of the sum and difference of two quantities is equal to the difference of their squares (Art. 103); therefore  $(\sqrt{a} + \sqrt{b})$  multiplied by  $(\sqrt{a} - \sqrt{b})$  =  $a - b$ , which is a rational quantity.

Therefore, the factor to employ as a multiplier is  $\sqrt{a} - \sqrt{b}$ .

OPERATION.

$$\begin{array}{r} \sqrt{a} + \sqrt{b} \\ \sqrt{a} + \sqrt{b} \\ \hline a + \sqrt{ab} \\ - \sqrt{ab} - b \\ \hline a - b, \text{ Ans.} \end{array}$$

2. What factor will rationalize  $\sqrt{x} - \sqrt{y}$ ?

ANALYSIS.—If the binomial  $\sqrt{x} - \sqrt{y}$  is multiplied by the same terms with the sign of the latter changed to +, we have

$(\sqrt{x} - \sqrt{y}) \times (\sqrt{x} + \sqrt{y}) = x - y$ .  
(Art. 103.) Therefore,  $\sqrt{x} + \sqrt{y}$  is the factor required. Hence, the

OPERATION.

$$\begin{array}{r} \sqrt{x} - \sqrt{y} \\ \sqrt{x} + \sqrt{y} \\ \hline x - y \\ \text{Ans. } \sqrt{x} + \sqrt{y} \end{array}$$

**RULE.**—*Multiply the binomial radical by the corresponding binomial with its connecting sign changed.*

3. What factor will rationalize  $x + 4\sqrt{9}$ ?

4. Rationalize  $\sqrt{9} - \sqrt{6}$ .

5. What factor will rationalize  $\sqrt{7} + \sqrt{a}$ ?

6. Rationalize  $6 - \sqrt{5}$ .

7. What factor will rationalize  $\sqrt{3a} - \sqrt{3b}$ ?

8. Rationalize  $\sqrt{a} - \sqrt{5}$ .

9. What factor will rationalize  $3\sqrt{a} + \sqrt{8}$ ?

10. What factor will rationalize  $4\sqrt{2a} - 5\sqrt{b}$ ?

## CASE III.

**318. To Change a Radical Fraction to one whose Numerator or Denominator is a Rational Quantity.**

1. Change  $\frac{a}{\sqrt{b}}$  to a rational denominator.

ANALYSIS.—Multiply both terms of the fraction by the denominator  $\sqrt{b}$ , and the

result is  $\frac{a\sqrt{b}}{b}$ , whose denominator is

rational. (Art. 167, Prin. 3, *note*.) Hence, the

OPERATION.

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

**RULE.**—*Multiply both terms of the fraction by such a factor as will make the required term rational.*

**NOTE.**—Since the product of the sum and difference of two quantities is equal to the difference of their squares, when the radical fraction is of the form  $\frac{x}{\sqrt{a} - \sqrt{b}}$ , if we multiply the terms by  $(\sqrt{a} + \sqrt{b})$ , we have  $a - b$  for the denominator. (Art. 103.)

2. Rationalize the denominator of  $\frac{3}{\sqrt{5}}$ .      *Ans.*  $\frac{3\sqrt{5}}{5}$ .

3. Rationalize the numerator of  $\frac{\sqrt{a}}{\sqrt{x}}$ .      *Ans.*  $\frac{a}{\sqrt{ax}}$ .

4. Rationalize the denominator of  $\frac{c}{\sqrt[3]{x}}$ .

5. Rationalize the denominator of  $\frac{a}{\sqrt[3]{c}}$ .

6. Rationalize the denominator of  $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$ .

7. Rationalize the denominator of  $\frac{x}{\sqrt{a} - \sqrt{b}}$ .

8. Rationalize the denominator of  $\frac{1}{1 + \sqrt{3}}$ .

9. Rationalize the denominator of  $\frac{\sqrt{3}}{3 - \sqrt{3}}$ .

318. How reduce a radical fraction to one whose numerator or denominator is a rational quantity? When the fractions contain compound quantities, what principle enters into their reduction?

# RADICAL EQUATIONS.

**319.** A *Radical Equation* is one in which the unknown quantity is under the radical sign.

## 320. To Solve a Radical Equation.

1. Given  $\sqrt{x} + 2 = 7$ , to find  $x$ .

ANALYSIS.—Transposing 2, we have,  
 $\sqrt{x} = 5$ . Since 5 is equal to the  $\sqrt{x}$ , it follows that the square of 5, or 25, must be the square of  $\sqrt{x}$ . Therefore,  $x = 25$ .

$$\sqrt{x} + 2 = 7$$

$$\sqrt{x} = 7 - 2 = 5$$

$$x = 5^2 = 25$$

2. Given  $2a + \sqrt{x} = 9a$ , to find  $x$ .

SOLUTION.—By the problem,  $2a + \sqrt{x} = 9a$

By transposing,  $\sqrt{x} = 7a$

By involution,  $x = 49a^2$

3. Given  $5\sqrt[3]{x+1} = 35$ , to find  $x$ .

SOLUTION.—By the problem,  $5\sqrt[3]{x+1} = 35$

Removing coefficient,  $\sqrt[3]{x+1} = 7$

Involving,  $x+1 = 343$

Transposing,  $x = 342$ . Hence, the

**RULE.**—*Involve both sides to a power of the same name as the root denoted by the radical sign.*

**NOTE.**—Before *involving* the quantities, it is generally best to clear of fractions, and transpose the terms, so that the quantities under the radical sign shall stand alone on one side of the equation.

Reduce the following radical equations:

4.  $a + \sqrt{x} + c = d$ .

8.  $\sqrt{2x+3} - 6 = 13$ .

5.  $\sqrt[3]{x+2} = 3$ .

9.  $\sqrt[3]{x-4} = 3$ .

6.  $3\sqrt{x-4} + 5 = 7\frac{1}{2}$ .

10.  $2\sqrt[4]{x-5} = 4$ .

7.  $3\sqrt{\frac{x}{4}} = 24$ .

11.  $5\sqrt{\frac{x}{7}} = 30$ .

319. What is a radical equation? 320. How solved? *Note.* What should be done before involving the quantities?

12. Given  $\frac{a + \sqrt{2ax + b}}{b} = b^2$ , to find  $x$ .

13. Reduce  $\sqrt{a^2 + \sqrt{x}} = \frac{3 + c}{\sqrt{(a^2 + \sqrt{x})}}$ .

ANALYSIS.—By removing the denominator the first member is squared. But  $x$  is still under the radical sign. This is removed by involving both members again.

OPERATION.

$$\sqrt{a^2 + \sqrt{x}} = \frac{3 + c}{\sqrt{(a^2 + \sqrt{x})}}$$

$$a^2 + \sqrt{x} = 3 + c$$

$$\text{Ans. } x = (3 + c - a^2)^2$$

14. Given  $\frac{y - ay}{\sqrt{y}} = \frac{\sqrt{y}}{y}$ , to find  $y$ .

15. Given  $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{(a^2 + x^2)}}$ , to find  $x$ .

NOTE.—If the equation has two radical expressions, connected with other terms by the signs + or −, it is advisable to transpose the terms so that one of the radicals shall stand alone on one side of the equation. By involving both members, one of the radicals becomes rational; and by repeating the operation, the other will also disappear.

16. Given  $\sqrt{4 + 5x} - \sqrt{3x} = 2$ , to find  $x$ .

Transposing,

$$\sqrt{4 + 5x} = \sqrt{3x} + 2$$

Involving,

$$4 + 5x = 4 + 4\sqrt{3x} + 3x$$

Transposing and dividing by 4,

$$\sqrt{3x} = \frac{x}{2}$$

Involving,

$$3x = \frac{x^2}{4}$$

Transposing and multiplying by 4,

$$x^2 = 12x$$

Hence,

$$x = 12, \text{ Ans.}$$

17. Given  $\sqrt{x + 12} = 2 + \sqrt{x}$ , to find  $x$ .

18. Given  $\sqrt{5} \times \sqrt{x + 2} = 2 + \sqrt{5x}$ , to find  $x$ .

19. Given  $\frac{\sqrt{x}}{x} = \frac{x - ax}{\sqrt{x}}$ , to find  $x$ .

## CHAPTER XV.

### QUADRATIC EQUATIONS.

**321.** Equations are divided into *different degrees*, as the first, second, third, etc., according to the powers of the unknown quantity contained in them.

An equation of the *First Degree* is called a *Simple Equation*, and contains only the *first* power of the unknown quantity.

An equation of the *Second Degree* is called a *Quadratic Equation*, and the highest power of the unknown quantity it contains is a *square*.

An equation of the *Third Degree* is called a *Cubic Equation*, and the highest power of the unknown quantity it contains is a *cube*.

An equation of the *Fourth Degree* is called a *Biquadratic*, etc.

**322.** *Quadratic Equations* are divided into *pure* and *affected*.

**323.** A *Pure Quadratic* contains the *square* only of the unknown quantity; as,  $x^2 = b$ .

**324.** An *Affected Quadratic* contains both the *first* and *second* powers of the unknown quantity; as,  $x^2 + ax = cd$ .

NOTES.—I. *Pure* quadratics are sometimes called *incomplete* equations; and *affected* quadratics, *complete* equations.

---

321. How are equations divided? What is an equation of the first degree? The second? Third? Fourth? 322. How are quadratic equations divided? 323. What is a pure quadratic? 324. An affected quadratic? *Note.* What are they sometimes called?

2. A *Complete Equation* contains every integral power of the unknown quantity from that which denotes its degree down to the zero power.

An *Incomplete Equation* is one which lacks one or more of these powers.

## PURE QUADRATICS.

**325.** Every *pure quadratic* may be reduced to the form

$$x^2 = a.$$

For, by transposition, etc., all the terms containing  $x^2$  can be reduced to one term, as  $bx^2$ ; and all the known quantities to one term, as  $c$ . Then will

$$bx^2 = c.$$

Dividing both members by  $b$ , and substituting  $a$  for the quotient of  $c \div b$ , the result is the form,

$$x^2 = a.$$

**326.** Pure quadratic equations have *two roots*, which are the *same* numerically, but have *opposite* signs. (Art. 293, *n*.)

Thus, the square of  $+a$  and of  $-a$  is equally  $a^2$ . Hence,

$$\sqrt{a^2} = \pm a.$$

**327.** To Solve a *Pure Quadratic Equation*.

1. Find the value of  $x$  in  $\frac{5x^2}{9} - 6 = \frac{x^2}{3} + 2$ .

SOLUTION.—Given  $\frac{5x^2}{9} - 6 = \frac{x^2}{3} + 2$

Clearing of fractions,  $5x^2 - 54 = 3x^2 + 18$

Transposing, etc.,  $2x^2 = 72$

Removing coefficient,  $x^2 = 36$

Extracting sq. root,  $x = \pm 6$ , *Ans.*

Substituting  $b$  for 2, and  $c$  for 72, in the third equation, we have the form,

$$bx^2 = c.$$

Removing coefficient, etc.,  $x^2 = a$ . Hence, the

**RULE.**—Reduce the given equation to the form  $x^2 = a$ , and extract the square root of both members. (Art. 296.)

Find the value of  $x$  in the following equations:

2.  $3x^2 - 5 = 70.$       10.  $2x^2 + 12 = 3x^2 - 37.$

3.  $9x^2 + 8 = 3x^2 + 62.$       11.  $7x^2 - 7 = 3x^2 + 9.$

4.  $5x^2 + 9 = 2x^2 + 57.$       12.  $a^2x^3 = a^4.$

5.  $6x^2 + 5 = 4x^2 + 55.$       13.  $(x + 2)^2 = 4x + 5.$

6.  $\frac{5x^2}{4} + 35 = 3x^2 + 7.$       14.  $x^2 - 1 = \frac{6x^2 - 12}{4}.$

7.  $\frac{2x^2 + 8}{10} = \frac{x^2 - 6}{10} + 5.$       15.  $\frac{x(2x + 9)}{30} = \frac{3x + 6}{10}.$

8.  $\frac{x}{4} = \frac{x}{2} - \frac{4}{x}.$       16.  $\frac{5}{4 - x} + \frac{5}{4 + x} = \frac{8}{3}.$

9.  $\frac{x}{2} + \frac{2}{x} = \frac{x}{3} + \frac{3}{x}.$       17.  $\frac{ax^2(a - 2)}{1 + x} = 1 - x.$

**328.** Radical equations, when cleared of radicals, often become pure quadratics.

18. Given  $\sqrt{x^2 + 11} = \sqrt{2x^2 - 5}$ , to find  $x$ .

**SOLUTION.**—Clearing of radicals,       $x^2 + 11 = 2x^2 - 5$   
Transposing and extracting root,       $x = \pm 4$

19. Given  $2\sqrt{x^2 - 5} = \frac{4x}{3}$ , to find  $x$ .

20. Given  $2\sqrt{x^2 - 4} = 4\sqrt{a^2 - 1}$ , to find  $x$ .

21. Given  $\sqrt{x + c} = \frac{d}{\sqrt{x - c}}$ , to find  $x$ .

22. Given  $\sqrt{\frac{5x^2 - 1}{x}} = \sqrt{x}$ , to find  $x$ .

23. Given  $\frac{b}{\sqrt{(x - a)}} = \sqrt{x + a}$ , to find  $x$ .

24. Given  $\frac{24}{\sqrt{x + 10}} = \sqrt{x - 10}$ , to find  $x$ .

---

327. What is the rule for the solution of pure quadratics? 328. What may radical equations become?

## PROBLEMS

1. The product of one-third of a number multiplied by one-fourth of it is 108. What is the number?
2. What number is that, the fourth part of whose square being subtracted from 25, leaves 9?
3. How many rods on one side of a square field whose area is 10 acres?
4. A gentleman exchanges a rectangular piece of land 50 rods long and 18 wide, for one of equal area in a square form. Required the length of one side of the square.
5. Find two numbers that are to each other as 2 to 5, and whose product is 360.
6. If the number of dollars which a man has be squared and 7 be subtracted, the remainder is 29. How much money has he?
7. Find a number whose eighth part multiplied by its fifth part and the product divided by 16, will give a quotient of 10.
8. The product of two numbers is 900, and the quotient of the greater divided by the less is 4. What are the numbers?
9. A merchant buys a piece of silk for \$40.50, and the price per yard is to the number of yards as 3 to 54. Required the number of yards and the price of each.
10. Find a number such that if 3 times the square be divided by 4 and the quotient be diminished by 12, the remainder will be 180.
11. A reservoir whose sides are vertical holds 266,112 gallons of water, is 6 feet deep, and square on the bottom. Required the length of one side, allowing 231 cubic inches to the gallon.
12. What number is that, to which if 10 be added, and from which if 10 be subtracted, the product of the sum and difference will be 156?

## AFFECTED QUADRATICS.\*

**329.** An *Affected Quadratic Equation* is one which contains the *first* and *second* powers of the unknown quantity; as,  $ax^2 + bx = c$ .

**330.** Every affected quadratic may be reduced to the form,  

$$x^2 \pm ax = b,$$
in which  $a$ ,  $b$ , and  $x$  may denote any quantity, either *positive* or *negative*, *integral* or *fractional*.

For, by transposition, etc., all the terms containing  $x^2$  can be reduced to one term, as  $cx^2$ ; also, those containing  $x$  can be reduced to one term, as  $dx$ ; and all containing the known quantities can be reduced to one term, as  $g$ . Then,  $cx^2 + dx = g$ .

Dividing both members by  $c$ , and substituting  $a$  for the quotient of  $d + c$ , and  $b$  for the quotient of  $g + c$ , we have,

$$x^2 + ax = b.$$

Take any numerical quadratic, as  $\frac{8x^2}{6} - \frac{2x}{3} - 8 = x^2 + \frac{4x}{6} - 4$ .

Clearing of fractions,  $8x^2 - 4x - 48 = 6x^2 + 4x - 24$

Transposing, etc.,  $2x^2 - 8x = 24$

Removing the coefficient,  $x^2 - 4x = 12$

Substituting  $a$  for 4, and  $b$  for 12 in the last equation, we have,

$$x^2 - ax = b. \text{ Hence,}$$

*All affected quadratics may be reduced to the general form,*

$$x^2 \pm ax = b.$$

**331.** The *First Member* of the general form of an affected quadratic equation, it will be seen, is a *Binomial*, but not a *Complete Square*. One term is *wanting* to make the square complete. (Art. 266, *note*.) The equation, therefore, cannot be solved in its present state.

329. What is an affected quadratic equation? 330. To what general form may every affected quadratic be reduced? 331. What is true of the first member of the general form of an affected quadratic?

\* *Quadratic*, from the Latin *quadrare*, to make square.

*Affected*, made up of *different powers*; from the Latin *ad* and *facio*, to make or join to,

**332.** There are *three* methods of completing the square and solving the equation.

### FIRST METHOD.

1. Given  $x^2 + 2ax = b$ , to find the value of  $x$ .

**ANALYSIS.**—The *first* and *third* terms of the square of a binomial are *complete* powers, and the *second* term is twice the *product* of their *roots*;

**OPERATION.**

$$\begin{aligned} x^2 + 2ax &= b \\ x^2 + 2ax + a^2 &= a^2 + b \\ x + a &= \pm \sqrt{a^2 + b} \\ x &= -a \pm \sqrt{a^2 + b} \end{aligned}$$

or the product of one of the roots into twice the other. (Art. 101.)

In the expression,  $x^2 + 2ax$ , the first term is a perfect square, and the second term  $2ax$  consists of the factors  $2a$  and  $x$ . But  $x$  is the root of the first term  $x^2$ ; therefore, the other factor  $2a$  must be *twice* the root of the third term which is required to complete the square. Hence, half of  $2a$ , or  $a$ , must be the root of the third term, and  $a^2$  the term itself. Therefore,  $x^2 + 2ax + a^2$  is the square of the first member completed.

But since we have added  $a^2$  to the *first* member of the equation, we must also add it to the *second*, to preserve the equality. Extracting the square root of both members, and transposing  $a$ , we have  $x = -a \pm \sqrt{a^2 + b}$ , the value sought. (Art. 297.)

2. What is the value of  $x$  in  $2x^2 + x = 64 - 7x$ ?

**ANALYSIS.**—Transposing  $-7x$  and removing the coefficient of  $x^2$ , we have the form  $x^2 + 4x = 32$ . But the first member,  $x^2 + 4x$ , is an incomplete square of a binomial.

**OPERATION.**

$$\begin{aligned} 2x^2 + x &= 64 - 7x \\ 2x^2 + 8x &= 64 \\ x^2 + 4x &= 32 \\ x^2 + 4x + 4 &= 36 \\ x + 2 &= \pm 6 \\ x &= -2 \pm 6 \\ \text{i. e., } x &= 4 \text{ or } -8 \end{aligned}$$

In order to complete the square, we add to it the square of half the coefficient of  $x$ . (Art. 266.) Now, having added 4 to one member of the equation, we must also add 4 to the other, to preserve their equality. Extracting the root of both, and transposing, we have  $x = 4$ , or  $-8$ . (Art. 297.)

NOTES.—I. Adding the *square of half the coefficient of the second term* to both members of the equation is called *completing the square*.

2. The first member of the fourth equation is the *square of a binomial*; therefore, its root is found by taking the roots of the *first and third terms*, which are perfect powers. (Art. 297.) From the process of squaring a binomial, it is obvious that the middle term ( $4x$ ) forms no part of the root. (Art. 266.)

**333.** From these illustrations we derive the following

RULE.—I. *Reduce the equation to the form,  $x^2 \pm ax = b$ .*

II. *Add to each member the square of half the coefficient of  $x$*

III. *Extract the square root of each, and reduce the resulting equation.*

3. Find the value of  $x$  in  $-2x^2 + 8ax = -6b$ .

SOLUTION.—By the problem,

$$-2x^2 + 8ax = -6b$$

Removing coefficient of  $x^2$ ,

$$-x^2 + 4ax = -3b$$

Making  $x^2$  positive (Art. 140, Prin. 3),  $x^2 - 4ax = 3b$

Completing square,

$$x^2 - 4ax + 4a^2 = 4a^2 + 3b$$

Extracting the root,

$$x - 2a = \pm \sqrt{4a^2 + 3b}$$

$$\therefore x = 2a \pm \sqrt{4a^2 + 3b}$$

4. Given  $x^2 + ax + bx = d$ , to find  $x$ .

OPERATION.

$$x^2 + ax + bx = d$$

$$x^2 + (a + b)x = d$$

$$x^2 + (a + b)x + \left(\frac{a + b}{2}\right)^2 = \left(\frac{a + b}{2}\right)^2 + d$$

$$x + \frac{a + b}{2} = \pm \sqrt{\left(\frac{a + b}{2}\right)^2 + d}$$

$$\therefore x = -\frac{a + b}{2} \pm \sqrt{\left(\frac{a + b}{2}\right)^2 + d}$$

ANALYSIS.—Factoring the terms which contain the first power of  $x$ , we have  $ax + bx = (a + b)x$ ; hence,  $(a + b)$  may be considered a compound coefficient of  $x$ . By adding the square of half this coefficient to both members, and extracting the root, the value of  $x$  is found.

5. Given
- $3x - 2x^2 = -9$
- , to find
- $x$
- .

SOLUTION.—By the problem,  $3x - 2x^2 = -9$ 

$$\text{Making } x^2 \text{ positive, etc.,} \quad x^2 - \frac{3x}{2} = \frac{9}{2}$$

$$\text{Completing square,} \quad x^2 - \frac{3x}{2} + \frac{9}{16} = \frac{9}{2} + \frac{9}{16} = \frac{81}{16}$$

$$\text{Extracting root,} \quad x - \frac{3}{4} = \pm \frac{9}{4}$$

$$\therefore x = \frac{3}{4} \pm \frac{9}{4}, \text{ i. e., } x = 3 \text{ or } -1\frac{1}{4}.$$

6. Given
- $3x^2 - 24x = -36$
- , to find
- $x$
- .

$$\text{Ans. } +6 \text{ or } +2.$$

NOTE.—The two roots of an affected quadratic may have the *same* or *different* signs. Thus, in the 6th and 12th examples they are the same; in the 1st, 2d, 3d, 4th, and 5th, they are different.

7. Given
- $5x^2 - 40x = 45$
- , to find
- $x$
- .

8. Given
- $x^2 - 6ax = d$
- , to find
- $x$
- .

9. Given
- $2x^2 + 2ax = 2(b + c)$
- , to find
- $x$
- .

SOLUTION.—Completing the square,  $x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} + b + c$ .

$$\text{Extracting root,} \quad x + \frac{a}{2} = \pm \sqrt{\frac{a^2}{4} + b + c}$$

$$\text{Transposing,} \quad x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b + c}$$

10. Given
- $2x^2 - 22x = 120$
- , to find
- $x$
- .

11. Given
- $x^2 - 140 = 13x$
- , to find
- $x$
- .

12. Find the value of
- $x$
- in
- $x^2 - 3x + 1 = 5x - 15$
- .

SOLUTION.—By the problem,  $x^2 - 3x + 1 = 5x - 15$ 

$$\text{Transposing,} \quad x^2 - 8x = -16$$

$$\text{Completing square,} \quad x^2 - 8x + 16 = 0$$

$$\text{Extracting root,} \quad x - 4 = 0$$

$$\therefore x = 4$$

NOTE.—In this equation, both the *signs* and the *numerical values* of the two roots are *alike*. Such equations are said to have *equal roots*.

---

*Note.*—What signs have the roots of an affected quadratic?

## SECOND METHOD.

**334.** When an affected quadratic equation has been reduced to the general form,

$$x^2 + ax = b,$$

its root may be obtained *without recourse* to completing the square.

1. Given  $x^2 + 8x = 65$ , to find  $x$ .

**ANALYSIS.**—After the square of an affected quadratic is completed and the root extracted, the root of the third term is transposed to the second member, by changing its sign. (Art. 204.)

**OPERATION.**

$$x^2 + 8x = 65$$

$$x = -4 \pm \sqrt{65 + 16}$$

$$\therefore x = -4 \pm 9$$

$$\text{i.e., } x = 5 \text{ or } -13$$

Now, if we prefix half the coefficient of  $x$ , with its sign changed, to plus or minus the square root of the second member increased by the square of half the coefficient of  $x$ , the second member of the equation will contain the *same combinations* of the *same terms*, as when the square is completed in the ordinary way. Hence, the

**RULE.**—*Prefix half the coefficient of  $x$ , with the opposite sign, to plus or minus the square root of the second member, increased by the square of half the coefficient of  $x$ .*

Solve the following equations:

2.  $3x^2 - 9x - 3 = 207.$

8.  $x^2 + 4ax = b.$

3.  $4x^2 + 12x + 5 = 45.$

9.  $3x^2 - 74 = 6x + 31.$

4.  $3x^2 - 14x + 15 = 0.$

10.  $x^2 + 13 = 6x.$

5.  $4x^2 - 9x = 28.$

11.  $(x - 2)(x - 1) = 20.$

6.  $\frac{x+2}{2x} + \frac{2x}{x+2} = 2.$

12.  $\frac{x+1}{x} + \frac{x}{x+1} = \frac{13}{6}.$

7.  $x^2 + \frac{ax}{b} - ab = d.$

13.  $x^2 - \frac{bx}{c} + ch = bd.$

## THIRD METHOD.

**335.** A third method of reducing an affected quadratic equation may be illustrated in the following manner:

1. Given  $ax^2 + bx = c$ , to find  $x$ .

**ANALYSIS.** — Multiply-  
ing the given equation by  
 $a$ , the coefficient of  $x^2$ , and  
by 4, the smallest square  
number, we have

$$4a^2x^2 + 4abx = 4ac,$$

the first term of which is  
an exact square, whose  
root is  $2ax$ . Factoring

the second term, we have  $4abx = 2(2ax \times b)$ . (Art. 119.)

As the factor  $2ax$  is the square root of  $4a^2x^2$ , it is evident that  $4a^2x^2$  may be regarded as the first term, and  $4abx$  the middle term of the square of a binomial. Since  $4abx$  is twice the product of this root  $2ax$  into  $b$ , it follows that  $b$  is the second term of the binomial; consequently,  $b^2$  added to both members will make the first a complete square, and preserve the equality. (Axiom 2.) Extracting the square root, transposing, etc., we have,

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}, \text{ the value of } x \text{ required.}$$

2. Given  $2x^2 + 3x = 27$ , to find  $x$ .

**SOLUTION.**—By the problem,

$$2x^2 + 3x = 27$$

Multiplying by 4 times coef. of  $x^2$ ,  $16x^2 + 24x = 216$

Adding square of 3, coef. of  $x$ ,  $16x^2 + 24x + 9 = 225$

Extracting root,

$$4x + 3 = \pm 15$$

Transposing,

$$4x = -3 \pm 15$$

$$\therefore x = 3 \text{ or } -4\frac{1}{4}.$$

**336.** From the preceding illustrations, we derive the

**RULE.**—I. Reduce the equation to the form,  $ax^2 \pm bx = c$ .

II. Multiply both members by 4 times the coefficient of  $x^2$ .

III. Add the square of the coefficient of  $x$  to each member, extract the root, and reduce the resulting equation.

---

336. What is the rule for the third method of reducing affected quadratics?

NOTES.—1. When the coefficient of  $x$  is an *even* number it is sufficient to multiply both members by the coefficient of  $x^2$ , and add to each the *square of half* the coefficient of  $x$ .

2. The *object* of multiplying the equation by the coefficient of  $x^2$  is to make the first term a *perfect square* without removing the coefficient. (Art. 251.)

3. The *reason* for multiplying by 4, is that it *avoids fractions* in completing the square, when the coefficient of  $x$  is an odd number. For, multiplying both members by 4, and adding the square of the entire coefficient of  $x$  to each, is the same in effect as adding the *square of half* the coefficient of  $x$  to each, and then clearing the equation of fractions by multiplying it by the denominator 4.

4. This method of completing the square is ascribed to the Hindoos.

3. Given  $3x^2 + 4x = 39$ , to find  $x$ . *Ans.* 3 or  $-4\frac{1}{3}$ .

Reduce the following equations:

4.  $x^2 - 30 = -x$ .

8.  $2x^2 - 6x = 8$ .

5.  $5x + 3x^2 = 2$ .

9.  $3x^2 + 5x = 42$ .

6.  $4x^2 - 7x - 2 = 0$ .

10.  $x^2 - 15x = -54$ .

7.  $5x^2 + 2x = 88$ .


11.  $9x^2 - 7x = 116$ .

**337.** The preceding methods are equally applicable to all classes of affected quadratics, but each has its *advantages* in particular problems.

The *first* is perhaps the *most natural*, being derived from the square of a binomial; but it necessarily *involves fractions*, when the coefficient of  $x$  is an *odd number*.

The *second* is the shortest, and is therefore a *favorite* with experts in algebra.

The advantage of the *third* is, that it always *avoids fractions* in completing the square.

 The student should exercise his judgment as to the method best adapted to his purpose.

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*Notes.* When the coefficient of  $x$  is an even number, how proceed? Object of multiplying by coefficient of  $x^2$ ? By 4?

## EXAMPLES.

Find the value of  $x$  in the following equations:

1.  $x^2 - 4x = -3$ .
2.  $x^2 - 5x = -4$ .
3.  $2x^2 - 7x = -3$ .
4.  $x^2 + 10x = 24$ .
5.  $6x^2 - 13x + 6 = 0$ .
6.  $14x - x^2 = 33$ .
7.  $x^2 - 3 = \frac{x-3}{6}$ .
8.  $\frac{5x^2}{7} + \frac{7x}{5} = -\frac{73}{140}$ .
9.  $\frac{16}{x} - \frac{100-9x}{4x^2} = 3$ .
10.  $\frac{a}{x} + \frac{x}{a} = \frac{2}{a}$ .
11.  $x^2 + 2mx = b^2$ .
12.  $x^2 + \frac{x}{8} = 1\frac{1}{8}$ .
13.  $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$ .
14.  $\frac{4x}{14-x} - \frac{x-1}{3x} = \frac{9x+7}{x}$ .
15.  $2\sqrt{x^2 - 4x - 1} = -4x$ .
16.  $\sqrt{x+5} + 6 = x + 5$ .
17.  $3x^2 - 7x - 20 = 0$ .
18.  $7x^2 - 160 = 3x$ .
19.  $2x^2 - 2x = 1\frac{1}{2}$ .
20.  $(x-2)(x-1) = 6$ .
21.  $4(x^2 - 1) = 4x - 1$ .
22.  $(2x-3)^2 = 8x$ .
23.  $3x - 2 = \frac{14}{x-1}$ .
24.  $4x - \frac{14-x}{x+1} = 14$ .
25.  $x^2 + \frac{3}{25} = \frac{4x}{5}$ .
26.  $x^2 + \frac{x}{2} = \frac{1}{2}$ .
27.  $x^2 - 2nx = m^2 - n^2$ .
28.  $\frac{9(b-a)}{x} = \frac{x-3a}{b}$ .

**338.** An Equation which contains but *two powers* of the unknown quantity, the *index* of one power being *twice* that of the other, is said to have the *Quadratic Form*.

The indices of these powers may be either *integral* or *fractional*.

Thus,  $x^4 - x^2 = 12$ ;  $x^{2n} + x^n = h$ ; and  $\sqrt[3]{x} - \sqrt[4]{x} = c$ , are equations of the quadratic form.

NOTE.—Equations of this character are sometimes called *trinomial* equations.

338. When has an equation the quadratic form? *Note.* What are such equations called?

**339.** Equations of the *quadratic form* may be solved by the rules for affected quadratics.

1. Given  $x^4 - 2x^2 = 8$ , to find  $x$ .

**SOLUTION**—By the problem,  $x^4 - 2x^2 = 8$

Completing square,  $x^4 - 2x^2 + 1 = 9$

Extracting square root,  $x^2 - 1 = \pm 3$

Transposing,  $x^2 = 4$  or  $-2$

Extracting square root again,  $x = \pm 2$ , or  $\pm \sqrt{-2}$

2. Given  $x^6 - 4x^3 = 32$ , to find  $x$ .

**SOLUTION**—By the problem,  $x^6 - 4x^3 = 32$

Completing square,  $x^6 - 4x^3 + 4 = 36$

Extracting square root,  $x^3 - 2 = \pm 6$

Transposing, etc.,  $x^3 = 8$  or  $-4$

Extracting cube root,  $x = 2$  or  $\sqrt[3]{-4}$

3. Given  $x^{2n} - 4bx^n = a$ , to find  $x$ .

**SOLUTION**—By the problem,  $x^{2n} - 4bx^n = a$

Completing square,  $x^{2n} - 4bx^n + 4b^2 = a + 4b^2$

Extracting square root,  $x^n - 2b = \pm \sqrt{a + 4b^2}$

Transposing,  $x^n = 2b \pm \sqrt{a + 4b^2}$

Extracting the  $n$ th root,  $x = \sqrt[n]{2b \pm \sqrt{a + 4b^2}}$

4. Given  $x^4 + 8 = 6x^2$ , to find  $x$ .

5. Given  $x^4 - 2x^2 = 3$ , to find  $x$ .

6. Given  $x^6 - 7x^3 = 0$ , to find  $x$ .

7. Given  $\frac{x^2}{2} + \frac{x}{4} = \frac{3}{32}$ , to find  $x$ .

8. Given  $\sqrt[3]{x^3} + \frac{1}{2}\sqrt[3]{x} = 1$ , to find  $x$ .

9. Given  $4x + 4\sqrt{x+2} = 7$ , to find  $x$ .

10. Given  $\frac{\sqrt{4x+20}}{4+\sqrt{x}} = \frac{4-\sqrt{x}}{\sqrt{x}}$ , to find  $x$ .

## PROBLEMS.

1. Find two numbers such that their sum is 12 and their product is 32.

2. A gentleman sold a picture for \$24, and the per cent lost was expressed by the cost of the picture. Find the cost.

NOTE.—Let  $x$  = the cost.

Then  $\frac{x}{100}$  = the per cent.

We now have  $x - x \times \frac{x}{100} = 24$ , to find the value of  $x$ .

3. The sum of two numbers is 10 and their product is 24. What are the numbers?

4. A person bought a flock of sheep for \$80; if he had purchased 4 more for the same sum, each sheep would have cost \$1 less. Find the number of sheep and the price of each.

5. Twice the square of a certain number is equal to 65 diminished by triple the number itself. Required the number.

6. A teacher divides 144 oranges equally among her scholars; if there had been 2 more pupils, each would have received one orange less. Required the number in the school.

7. A father divides \$50 between his two daughters, in such a proportion that the product of their shares is \$600. What did each receive?

8. Find two numbers whose sum is 100 and their product 2400.

9. The fence enclosing a rectangular field is 128 rods long, and the area of the field is 1008 square rods. What are its length and breadth?

10. A colonel arranges his regiment of 1600 men in a solid body, so that each rank exceeds the file by 60 soldiers. How many does he place in rank and file?

11. A drover buys a number of lambs for \$50 and sells them at \$5.50 each, and thus gains the cost of one lamb. Required the number of lambs.

12. The sum of two numbers is 4 and the sum of their reciprocals is 1. What are the numbers?

13. The sum of two numbers is 5 and the sum of their cubes 65. What are the numbers?

14. The length of a lot is 1 yard longer than the width and the area is 3 acres. Find the length of the sides.

15. A and B start together for a place 300 miles distant; A goes 1 mile an hour faster than B, and arrives at his journey's end 10 hours before him. Find the rate per hour at which each travels.

16. A and B distribute \$1200 each among a certain number of persons. A relieves 40 persons more than B, and B gives to each person \$5 more than A. Required the number relieved by each.

17. Divide 48 into two such parts that their product may be 252.

18. Two girls, A and B, bought 10 lemons for 24 cts., each spending 12 cents; A paid 1 cent more apiece than B: how many lemons did each buy?

19. Find the length and breadth of a room the perimeter of which is 48 feet, the area of the floor being as many square feet as 35 times the difference between the length and breadth.

20. In a peach orchard of 180 trees there are three more in a row than there are rows. How many rows are there, and how many trees in each?

21. Find the number consisting of two digits whose sum is 7, and the sum of their squares is 29.

22. The expenses of a picnic amount to \$10, and this sum could be raised if each person in the party should give 30 cts. more than the number in the party. How many compose the party?

23. Find two numbers the product of which is 120, and if 2 be added to the less and 3 subtracted from the greater, the product of the sum and remainder will also be 120.

24. Divide 36 into two such parts that their product shall be 80 times their difference.

25. The sum of two numbers is 75 and their product is to the sum of their squares as 2 to 5. Find the numbers.

26. Divide 146 into two such parts that the difference of their square roots may be 6.

27. The fore-wheel of a carriage makes sixty revolutions more than the hind-wheel in going 3600 feet; but if the circumference of each wheel were increased by three feet, it would make only forty revolutions more than the hind-wheel in passing over the same distance. What is the circumference of each wheel?

28. Find two numbers whose difference is 16 and their product 36.

29. What two numbers are those whose sum is  $1\frac{1}{3}$  and the sum of their reciprocals  $3\frac{1}{3}$ ?

30. Find two numbers whose difference is 15, and half their product is equal to the cube of the less number.

31. A lady being asked her age, said, If you add the square root of my age to half of it, and subtract 12, the remainder is nothing. What is her age?

32. The perimeter of a field is 96 rods, and its area is equal to 70 times the difference of its length and breadth. What are its dimensions?

33. The product of the ages of A and B is 120 years. If A were 3 years younger and B 2 years older, the product of their ages would still be 120. How old is each?

34. A man bought 80 pounds of pepper, and 36 pounds of saffron, so that for 8 crowns he had 14 pounds of pepper more than of saffron for 26 crowns; and the amount he laid out was 188 crowns. How many pounds of pepper did he buy for 8 crowns?

## SIMULTANEOUS QUADRATIC EQUATIONS.

## TWO UNKNOWN QUANTITIES.

**340.** A *Homogeneous Equation* is one in which the *sum* of the *exponents* of the unknown quantities is the same in every term which contains them. \*

Thus,  $x^2 - y^2 = 7$ , and  $x^2 - xy + y^2 = 13$ , are each homogeneous.

**341.** A *Symmetrical Equation* is one in which the unknown quantities are involved to the same degree.

Thus,  $x^2 + y^2 = 34$ , and  $x^2y - xy^2 = 34$ , are each symmetrical.

**342.** *Simultaneous Quadratic Equations* containing two unknown quantities, in general involve the principles of *Biquadratic* equations, which belong to the higher departments of Algebra.

There are three classes of examples, however, which may be solved by the *rules of quadratics*.

- 1st. When one equation is *quadratic*, and the other *simple*.
- 2d. When both equations are *quadratic* and *homogeneous*.
- 3d. When each equation is *symmetrical*.

**343.** To Solve Simultaneous Equations consisting of a Quadratic and a Simple Equation.

1. Given  $x^2 + y^2 = 13$ , and  $x + y = 5$ , to find  $x$  and  $y$ .

SOLUTION.—By the problem,  $x^2 + y^2 = 13$  (1)

“ “  $x + y = 5$  (2)

By transposition,  $x = 5 - y$  (3)

Squaring each side of (2) (Art. 102),  $x^2 = 25 - 10y + y^2$  (4)

Substituting (4) in (1),  $25 - 10y + y^2 + y^2 = 13$  (5)

Uniting and transposing,  $2y^2 - 10y = -12$  (6)

Comp. sq. (Art. 336, note),  $4y^2 - 20y + 25 = -24 + 25$  (7)

Extracting root,  $2y - 5 = \pm 1$

$\therefore y = 3$  or  $2$ .

Substituting value of  $y$  in (3),  $x = 2$  or  $3$ . Hence, the

**RULE.**—*Find the value of one of the unknown quantities in the simple equation by transposition, and substitute this value in the quadratic equation.* (Arts. 221, 223.)

Solve the following equations:

$$2. \quad x^2 + y^2 = 25,$$

$$x + y = 7.$$

$$3. \quad x^2 + y^2 = 74,$$

$$x + y = 12.$$

$$4. \quad x^2 - y^2 = 28,$$

$$x - y = 2.$$

$$5. \quad x^2 + y^2 = 244,$$

$$y - x = 2.$$

$$6. \quad 3x^2 - y^2 = 251,$$

$$x + 4y = 38.$$

$$7. \quad 8x^2 + 5y^2 = 728,$$

$$6y - x = 15.$$

### 344. To Solve Simultaneous Equations which are both Quadratic and Homogeneous.

8. Given  $x^2 + xy = 40$ , and  $y^2 + xy = 24$ , to find  $x$  and  $y$ .

**SOLUTION.**—By the problem,

$$x^2 + xy = 40 \quad (1)$$

“ “

$$y^2 + xy = 24 \quad (2)$$

Let

$$x = py \quad (3)$$

Substituting  $py$  in (1),

$$p^2y^2 + py^2 = 40 \quad (4)$$

“ “ (2),

$$y^2 + py^2 = 24 \quad (5)$$

Factoring, etc., (4),

$$y^2 = \frac{40}{p^2 + p} \quad (6)$$

“ “ (5),

$$y^2 = \frac{24}{1 + p} \quad (7)$$

Equating (6) and (7),

$$\frac{40}{p^2 + p} = \frac{24}{1 + p} \quad (8)$$

Clearing of fractions,

$$5 + 5p = 3p^2 + 3p \quad (9)$$

Transposing, etc.,

$$3p^2 - 2p = 5 \quad (10)$$

Comp. sq., 3d meth. (Art. 336),  $9p^2 - 6p + 1 = 15 + 1 = 16$  (11)

Extracting root,

$$3p - 1 = \pm 4 \quad (12)$$

Transposing,

$$3p = 1 \pm 4$$

Dropping the negative value,

$$p = \frac{5}{3}$$

Substituting value of  $p$  in (7),

$$y^2 = 24 + (1 + \frac{5}{3}) = 9$$

Extracting root,

$$y = \pm 3$$

Substituting value of  $p$  and  $y$  in (3),

$$x = \frac{5}{3} \times \pm 3 = \pm 5.$$

Hence, the

**RULE.—I.** *For one of the unknown quantities substitute the product of the other into an auxiliary quantity, and then find the value of this auxiliary quantity.*

**II.** *Find the values of the unknown quantities by substituting the value of the auxiliary quantity in one of the equations least involved.*

**NOTE.**—An auxiliary quantity is one introduced to aid in the solution of a problem, as  $p$  in the above operation.

$$\begin{array}{l} 9. \text{ Given } x + y = 9, \\ \text{And } x^2 + y^2 = 189, \end{array} \left. \vphantom{\begin{array}{l} 9. \\ \text{And} \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 10. \text{ Given } x - y = 2, \\ \text{And } x^2 - y^2 = 98, \end{array} \left. \vphantom{\begin{array}{l} 10. \\ \text{And} \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 11. \text{ Given } 3x^2 - 7y^2 = -1, \\ \text{And } 4xy = 24, \end{array} \left. \vphantom{\begin{array}{l} 11. \\ \text{And} \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 12. \text{ Given } x^2 - xy + y^2 = 19, \\ \text{And } xy = 15, \end{array} \left. \vphantom{\begin{array}{l} 12. \\ \text{And} \end{array}} \right\} \text{ to find } x \text{ and } y.$$

### 345. To Solve Simultaneous Quadratic Equations when each Equation is Symmetrical.

$$13. \text{ Given } x + y = 9, \text{ and } xy = 20, \text{ to find } x \text{ and } y.$$

SOLUTION.—By the problem,	$x + y = 9$	(1)
" " "	$xy = 20$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = 81$	(3)
Multiplying (2) by 4,	$4xy = 80$	(4)
Subtracting (4) from (3),	$x^2 - 2xy + y^2 = 1$	(5)
Extracting sq. root of (5),	$x - y = \pm 1$	(6)
Bringing down (1),	$x + y = 9$	
Adding (1) and (6),	$2x = 10 \text{ or } 8$	(7)
Removing coefficient,	$x = 5 \text{ or } 4$	
Substituting value of $x$ in (1),	$y = 4 \text{ or } 5$	

**NOTES.—I.** The values of  $x$  and  $y$  in these equations are not equal, but interchangeable; thus, when  $x = 5, y = 4$ ; and when  $x = 4, y = 5$ .

344. How solve equations which are both quadratic and homogeneous? *Note.* What is an auxiliary quantity?

2. The solution of this class of problems *varies* according to the given equations. Consequently, no specific rules can be given that will meet every case. But judgment and practice will readily supply expedients. Thus,

I. When the sum and product are given. (Ex. 13, 15.)

*Find the difference and combine it with the sum.* (Art. 224.)

II. When the difference and product are given. (Ex. 16.)

*Find the sum and combine it with the difference.*

III. When the sum and difference of the same powers are given. (Ex. 14, 17.)

*Combine the two equations by addition and subtraction.*

IV. When the members of one equation are multiples of the other. (Ex. 18.)

*Divide one by the other, and then reduce the resulting equation.*

$$\begin{array}{l} 14. \text{ Given } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5, \quad (1) \\ \text{And } x^{\frac{1}{2}} - y^{\frac{1}{2}} = 1, \quad (2) \end{array} \left. \vphantom{\begin{array}{l} 14. \text{ Given } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5, \\ \text{And } x^{\frac{1}{2}} - y^{\frac{1}{2}} = 1, \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{ll} \text{SOLUTION.} - \text{Adding (1) and (2), and dividing,} & x^{\frac{1}{2}} = 3 \\ \text{Involving,} & x = 27 \\ \text{Subtracting (2) from (1), etc.,} & y^{\frac{1}{2}} = 2 \\ \text{Involving,} & y = 8 \end{array}$$

$$\begin{array}{l} 15. \text{ Given } x + y = 27, \\ \text{And } xy = 180, \end{array} \left. \vphantom{\begin{array}{l} 15. \text{ Given } x + y = 27, \\ \text{And } xy = 180, \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 16. \text{ Given } x - y = 14, \\ \text{And } xy = 147, \end{array} \left. \vphantom{\begin{array}{l} 16. \text{ Given } x - y = 14, \\ \text{And } xy = 147, \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 17. \text{ Given } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 7, \\ \text{And } x^{\frac{1}{2}} - y^{\frac{1}{2}} = 3, \end{array} \left. \vphantom{\begin{array}{l} 17. \text{ Given } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 7, \\ \text{And } x^{\frac{1}{2}} - y^{\frac{1}{2}} = 3, \end{array}} \right\} \text{ to find } x \text{ and } y.$$

$$\begin{array}{l} 18. \text{ Given } x^2y^2 + x^2y^3 = 12, \\ \text{And } x^2y + xy^3 = 6, \end{array} \left. \vphantom{\begin{array}{l} 18. \text{ Given } x^2y^2 + x^2y^3 = 12, \\ \text{And } x^2y + xy^3 = 6, \end{array}} \right\} \text{ to find } x \text{ and } y.$$

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NOTE.—What is true of the solution of simultaneous quadratics? When the sum and product are given, how proceed? When the difference and product? When the sum and difference of the same powers are given? When the members of one equation are multiples of the other?

## PROBLEMS.

1. The difference of two numbers is 4, and the difference of their cubes 448. What are the numbers?
2. A man is one year older than his wife, and the product of their respective ages is 930. What is the age of each?
3. Required two numbers whose sum multiplied by the greater is 180, and whose difference multiplied by the less is 16.
4. In an orchard of 1000 trees, the number of rows exceeds the number of trees in each row by 15. Required the number of rows and the number of trees in each row.
5. The area of a rectangular garden is 960 square yards, and the length exceeds the breadth by 16 yards. Required the dimensions.
6. Subtract the sum of two numbers from the sum of their squares, and the remainder is 78; the product of the numbers increased by their sum is 39. What are the numbers?
7. Find two numbers whose sum added to the sum of their squares is 188, and whose product is 77.
8. A surveyor lays out a piece of land in a rectangular form, so that its perimeter is 100 rods, and its area 589 square rods. Find the length and breadth.
9. Required two numbers whose product is 28, and the sum of their squares 65.
10. A regiment of soldiers consisting of 1154 men is formed into two squares, one of which has 2 more men on a side than the other. How many men are on a side of each of the squares?
11. Required two numbers whose product is 3 times their sum, and the sum of their squares 160.
12. What two numbers are those whose product is 6 times their difference, and the sum of their squares 13?

## CHAPTER XVII.

### RATIO AND PROPORTION.

**346.** *Ratio* is the relation which one quantity bears to another with respect to magnitude.

**347.** The *Terms of a Ratio* are the quantities compared. The first is called the *Antecedent*, the second the *Consequent*,\* and the two together, a *Couplet*.

**348.** The *Sign* of ratio is a *colon* : † placed between the two quantities compared.

Ratio is also denoted by placing the *consequent under the antecedent*, in the form of a *fraction*.

Thus, the ratio of *a* to *b* is written,  $a : b$ , or  $\frac{a}{b}$ .

**349.** The *Measure* or *Value* of a ratio is the *quotient* of the antecedent divided by the consequent, and is equal to the value of the *fraction* by which it is expressed.

Thus, the measure or value of  $8 : 4$  is  $8 \div 4 = 2$ .

NOTE.—That quantities may have a *ratio* to each other, they must be so far of the same nature, that one can properly be said to be *equal* to, or *greater*, or *less* than the other.

Thus, a foot has a ratio to a *yard*, but not to an *hour*, or a *pound*.

**350.** A *Simple Ratio* is one which has but *two terms*; as,  $a : b$ ,  $8 : 4$ .

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346. What is ratio? 347. What are the terms of a ratio? 348. The sign? How also is ratio denoted? 349. The measure or value? Note. What quantities have a ratio to each other? 350. What is a simple ratio?

\* *Antecedent*, Latin *ante*, before, and *cedere*, to go, to *precede*.

*Consequent*, Latin *con*, and *sequi*, to follow.

† The sign of ratio : is derived from the sign of division +, the horizontal line being dropped.

**351. A Compound Ratio** is the product of *two or more simple ratios*.

Thus,  $4 : 2$  } are each simple  
 $9 : 3$  } ratios.

But  $4 \times 9 : 2 \times 3$   
 is a compound ratio.

NOTE.—The *nature* of compound ratios is the same as that of simple ratios. They are so called to denote their *origin*, and are usually expressed by writing the corresponding terms of the simple ratios one under another, as above.

**352. A Direct Ratio** arises from dividing the antecedent by the consequent.

**353. An Inverse\* or Reciprocal Ratio** arises from dividing the consequent by the antecedent, and is the same as the ratio of the *reciprocals* of the two numbers compared.

Thus, the direct ratio of  $a$  to  $ab = \frac{a}{ab}$ , or  $\frac{1}{b}$ , and that of  $4$  to  $12 = \frac{4}{12}$ , or  $\frac{1}{3}$ .

The inverse ratio of  $a$  to  $ab = \frac{ab}{a}$ , or  $b$ ; of  $4$  to  $12 = \frac{12}{4}$ , or  $3$ .  
 It is the same as the ratio of the reciprocals,  $\frac{1}{a}$  to  $\frac{1}{ab}$ , and  $\frac{1}{4}$  to  $\frac{1}{12}$ .

NOTE.—A *reciprocal* ratio is expressed by inverting the *fraction* which expresses the *direct* ratio. When the *colon* is used, it is expressed by *inverting the order* of the terms.

**354.** The ratio between *two fractions* which have a common denominator, is the same as the *ratio* of their *numerators*.

Thus, the ratio of  $\frac{2}{3} : \frac{1}{3}$  is the same as  $6 : 3$ .

NOTE.—When the fractions have *different denominators*, reduce them to a *common denominator*; then compare their numerators. (Art. 175.)

**355. A Ratio of Equality** is one in which the quantities compared are *equal*, and its *value* is a unit or 1.

351. What is a compound ratio? Note. Why so called? 352. What is a direct ratio? 353. A reciprocal? 355. What is a ratio of equality?

\* *Inverse*, from the Latin *in* and *verto*, to turn upside down, to invert.

**356.** A *Ratio of Greater Inequality* is one whose antecedent is *greater* than its consequent, and its value is *greater* than 1.

**357.** A *Ratio of Less Inequality* is one whose antecedent is *less* than its consequent, and its value is *less* than 1.

**358.** A *Duplicate Ratio* is the *square* of a *simple* ratio. It arises from multiplying a simple ratio into itself, or into another *equal* ratio.

**359.** A *Triplicate Ratio* is the *cube* of a *simple* ratio, and is the product of *three equal* ratios.

Thus, the *duplicate* ratio of  $a$  to  $b$  is  $a^2 : b^2$ .

The *triplicate* ratio of  $a$  to  $b$  is  $a^3 : b^3$ .

**360.** A *Subduplicate Ratio* is the *square root* of a simple ratio.

**361.** A *Subtriplicate Ratio* is the *cube root* of a simple ratio.

Thus, the *subduplicate* ratio of  $x$  to  $y$  is  $\sqrt{x} : \sqrt{y}$ .

The *subtriplicate* ratio of  $x$  to  $y$  is  $\sqrt[3]{x} : \sqrt[3]{y}$ , etc.

**362.** Since ratio may be expressed in the form of a *fraction*, it follows that changes made in its terms have the same effect on its value, as *like changes* in the terms of a fraction. (Art. 167.) Hence, the following

## PRINCIPLES.

- |   |   |                                 |
|---|---|---------------------------------|
| 1°. <i>Multiplying the antecedent, or</i> | } | <i>Multiplies the ratio.</i>    |
| <i>Dividing the consequent,</i>           |   |                                 |
| 2°. <i>Dividing the antecedent, or</i>    | } | <i>Divides the ratio.</i>       |
| <i>Multiplying the consequent,</i>        |   |                                 |
| 3°. <i>Multiplying or dividing both</i>   | } | <i>Does not alter the value</i> |
| <i>terms by the same quantity,</i>        |   |                                 |
|   |   | <i>of the ratio.</i>            |

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356. Of greater inequality? 357. Of less inequality? 358. A duplicate ratio?  
 359. Triplicate? 360. Subduplicate? 361. Subtriplicate? 362. Name Principle 1.  
 Principle 2. Principle 3.

EXAMPLES.

1. What is the ratio of 4 yards to 4 feet?

SOLUTION. 4 yards = 12 feet; and the ratio of 12 ft. to 4 ft. is 3

2. What is the ratio of  $6x^2$  to  $2x$ ? *Ans.*  $3x$ .

3. What is the ratio of 40 square rods to an acre?

4. What is the ratio of 1 pint to a gallon?

5. What is the ratio of 64 rods to a mile?

6. What is the ratio of  $8a^2$  to  $4a$ ?

7. What is the ratio of  $15abc$  to  $5ab$ ?

8. What is the ratio of \$5 to 50 cents?

9. What is the ratio of 75 cents to \$6?

10. What is the ratio of 35 quarts to 35 gallons?

11. What is the ratio of  $2a^2$  to  $4a$ ?

12. What is the ratio of  $x^2 - y^2$  to  $x + y$ ?

13. What is the compound ratio of 9 : 12 and 8 : 15?

SOLUTION.  $9 \times 8 = 72$ , and  $12 \times 15 = 180$ . Now  $72 \div 180 = \frac{2}{5}$ , *Ans.*  
Or,  $9 : 12 = \frac{3}{4}$ , and  $8 : 15 = \frac{8}{15}$ . Now  $\frac{3}{4} \times \frac{8}{15} = \frac{2}{5} = \frac{2}{5}$ , *Ans.*

14. What is the compound ratio of 8 : 15 and 25 : 30?

15. What is the compound ratio of  $a : b$  and  $2b : 3ax$ ?

16. Reduce the ratio of 9 to 45 to the lowest terms.

SOLUTION.  $9 : 45 = \frac{1}{5}$ , and  $\frac{1}{5} = \frac{1}{5}$ , *Ans.*

17. Reduce the ratio of 24 to 96 to the lowest terms.

18. Reduce the ratio of 144 to 1728 to the lowest terms.

19. What kind of ratio is 25 to 25?

20. What kind of a ratio is  $ab : ab$ ?

21. What kind of ratio is 35 to 7?

22. What kind of ratio is 6 to 48?

23. Which is the greater, the ratio of 15 : 9, or 38 : 19?

24. Which is the greater, the ratio of 8 : 25, or  $\sqrt{4} : \sqrt{25}$ ?

25. If the antecedent of a couplet is 56, and the ratio 8 what is the consequent?

26. If the consequent of a couplet is 7, and the ratio 14, what is the antecedent?

## PROPORTION.

**363.** *Proportion* is an equality of ratios.

Thus, the ratio  $8 : 4 = 6 : 3$ , is a proportion. That is,

Four quantities are in *proportion*, when the *first* is the *same multiple* or *part* of the *second* that the *third* is of the *fourth*.

**364.** The *Sign of Proportion* is a double colon  $::$ ,\* or the sign  $=$ . Thus,

The equality between the ratio of  $a$  to  $b$  and  $c$  to  $d$  is expressed by

$$a : b :: c : d, \text{ or by } \frac{a}{b} = \frac{c}{d}$$

The former is read, " $a$  is to  $b$  as  $c$  is to  $d$ ;" the latter, " $b$  is contained in  $a$  as many times as  $d$  is contained in  $c$ ."

NOTE.—Each ratio is called a *couplet*, and each term a *proportional*.

**365.** The *Terms* of a proportion are the quantities compared. The *first* and *fourth* are called the *extremes*, the *second* and *third* the *means*.

**366.** In every proportion there must be at least *four terms*; for the equality is between *two* or *more* ratios, and each ratio has *two terms*.

**367.** A proportion may, however, be formed from *three* quantities, for one of the quantities may be *repeated*, so as to form *two terms*; as,  $a : b :: b : c$ .

NOTE.—Care should be taken not to confound *proportion* with *ratio*. In common discourse, these terms are often used indiscriminately. Thus, it is said, "The income of one man bears a greater proportion to his capital than that of another," etc. But these are loose expressions.

In a simple ratio there are but *two terms*, an antecedent and a consequent; whereas, in a proportion there must at least be *four terms*. (Arts. 350, 366.)

363. What is proportion? 364. The sign of proportion? Note. What is each ratio called? 365. What are the terms of a proportion? 366. How many terms in every proportion? 367. How form a proportion from three quantities?

\* The sign  $::$  is derived from the sign of equality  $=$ , the four points being the terminations of the lines.

Again, one *ratio* may be *greater* or *less* than another, but one *proportion* is neither *greater* nor *less* than another. For equality does not admit of *degrees*. In scientific investigations, this distinction should be carefully observed.

**368.** A *Mean Proportional* between two quantities is the *middle term* or *quantity repeated*, in a proportion formed from *three quantities*.

**369.** A *Third Proportional* is the *last term* of a proportion having three quantities.

Thus, in the proportion  $a : b :: b : c$ ,  $b$  is a mean proportional, and  $c$  a third proportional.

**370.** A *Direct Proportion* is an equality between two *direct ratios*; as,  $a : b :: c : d$ ,  $3 : 6 :: 4 : 8$ .

**371.** An *Inverse* or *Reciprocal Proportion* is an equality between a *direct* and *reciprocal ratio*; as,

$$8 : 4 :: \frac{1}{4} : \frac{1}{8}.$$

**372.** *Analogous Terms* are the antecedent and consequent of the same couplet.

**373.** *Homologous Terms* are either two antecedents or two consequents.

## PROPOSITIONS.

**374.** A *Proposition* is the statement of a truth to be proved, or of an operation to be performed.

Propositions are of two kinds, *theorems* and *problems*.

**375.** A *Theorem* is something to be proved.

**376.** A *Problem* is something to be done.

**377.** A *Corollary* is a principle *inferred* from a preceding proposition.

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368. What is a mean proportional? 369. What is a third proportional? 370. A direct proportion? 371. An inverse or reciprocal proportion? 372. What are analogous terms? 373. Homologous? 374. What is a proposition? How divided? 375. What is a theorem? 376. A problem? 377. A corollary?

378. The more important theorems in proportion are the following:

### THEOREM I.

*If four quantities are proportional, the product of the extremes is equal to the product of the means.*

Let  $a : b :: c : d$

By Art. 363,  $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions,  $ad = bc$ .

#### VERIFICATION BY NUMBERS.

Given,  $2 : 4 :: 8 : 16$ ; and  $2 \times 16 = 4 \times 8$ .

COR.—The *relation* of the four terms of a proportion to each other is such, that if *any three* of them are given, the *fourth* may be found.

Thus, since  $ad = bc$ , it follows that

$a = bc \div d$ ,  $b = ad \div c$ ,  $c = ad \div b$ , and  $d = bc \div a$ . (Ax. 5.)

NOTES.—1. The rule of *Simple Proportion* in Arithmetic is founded upon this principle, and its operations are easily proved by it.

2. This theorem furnishes a very simple test for determining whether any four quantities are proportional. We have only to multiply the extremes together, and the means.

### THEOREM II.

*If three quantities are proportional, the product of the extremes is equal to the square of the mean.*

Let  $a : b :: b : c$

By Art. 363,  $\frac{a}{b} = \frac{b}{c}$

Clearing of fractions,  $ac = b^2$ .

Again,  $9 : 6 :: 6 : 4$ , and  $4 \times 9 = 6^2$ .

COR.—A *mean proportional* between *two quantities* is equal to the *square root* of their product.

### THEOREM III.

*If the product of two quantities is equal to the product of two others, the four quantities are proportional; the factors of either product being taken for the extremes, and the factors of the other for the means.*

$$\begin{array}{ll} \text{Let} & ad = bc \\ \text{Dividing by } bd, & \frac{a}{b} = \frac{c}{d} \\ \text{Or, by Art. 363,} & a : b :: c : d \\ \text{Again, } 4 \times 6 = 3 \times 8, \text{ and} & 4 : 3 :: 8 : 6. \end{array}$$

### THEOREM IV.

*If four quantities are proportional, they are proportional when the means are inverted.*

$$\begin{array}{ll} \text{Let } a : b :: c : d, \text{ then} & a : c :: b : d \\ \text{For, by Art. 363,} & \frac{a}{b} = \frac{c}{d} \\ \text{Multiplying by } \frac{b}{c}, & \frac{a}{c} = \frac{b}{d} \\ \text{Or,} & a : c :: b : d \\ \text{Again, } 3 : 6 :: 4 : 8, \text{ and } 3 : 4 :: 6 : 8. & (\text{Th. 1.}) \end{array}$$

NOTE.—This change in the order of the means is called “*Alternation*.”

### THEOREM V.

*If four quantities are proportional, they are proportional when the terms of each couplet are inverted.*

$$\begin{array}{ll} \text{Le. } a : b :: c : d, \text{ then} & b : a :: d : c \\ \text{By Theorem 1,} & ad = bc \\ \text{By Theorem 3,} & b : a :: d : c. \\ \text{Again, } 6 : 2 :: 15 : 5, \text{ then} & 2 : 6 :: 5 : 15. (\text{Th. 1.}) \end{array}$$

COR.—If the *extremes* are inverted, or the *order* of the terms, the quantities will be proportional.

NOTES.—1. If the terms of only *one* of the couplets are inverted, the proportion becomes *reciprocal*.

2. The change in the order of the terms of each couplet is called "*Inversion*."

3. This proposition supposes the quantities compared to be of the same kind. Thus, a line has no relation to weight. (Art. 349, *note*.)

### THEOREM VI.

*If four quantities are proportional, two analogous or two homologous terms may be multiplied or divided by the same quantity without destroying the proportion.*

Let  $a : b :: c : d$   
 Multiplying analogous terms,  $am : bm :: c : d$   
 and  $a : b :: cm : dm$

For,  $\frac{a}{b} = \frac{c}{d}$

Hence, (Art. 362, Prin. 3),  $\frac{am}{bm} = \frac{c}{d}$ , and  $\frac{a}{b} = \frac{cm}{dm}$

Multiplying homologous terms,  $am : b :: cm : d$

And  $a : bm :: c : dm$ .

Hence, (Ax. 4, 5),  $\frac{am}{b} = \frac{cm}{d}$ , and  $\frac{a}{bm} = \frac{c}{dm}$

Dividing analogous terms,  $\frac{a}{m} : \frac{b}{m} :: c : d$ ,

and  $a : b :: \frac{c}{m} : \frac{d}{m}$

Dividing homologous terms,  $\frac{a}{m} : b :: \frac{c}{m} : d$

and  $a : \frac{b}{m} :: c : \frac{d}{m}$

Clearing of fractions (Th. 1),  $ad = bc$

COR.—All the terms of a proportion may be multiplied or divided by the same quantity without destroying the proportion.

NOTES.—1. When the *homologous* terms are multiplied or divided, both ratios are *equally increased* or *diminished*.

2. When the *analogous* terms are multiplied or divided, the ratios are *not altered*.

### THEOREM VII.

*If four quantities are proportional, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.*

Let  $a : b :: c : d$ , then  $a + b : b :: c + d : d$

For,  $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member,  $\frac{a}{b} + 1 = \frac{c}{d} + 1$ . (Ax. 2.)

Incorporating 1,  $\frac{a+b}{b} = \frac{c+d}{d}$

Therefore (Art. 363),  $a + b : b :: c + d : d$

Again,  $4 : 2 :: 6 : 3$ , then  $4 + 2 : 2 :: 6 + 3 : 3$

NOTE.—This combination is sometimes called "*Composition*."

### THEOREM VIII.

*If four quantities are proportional, the difference of the first and the second is to the second, as the difference of the third and fourth is to the fourth.*

Let  $a : b :: c : d$ , then  $a - b : b :: c - d : d$

For,  $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member,  $\frac{a}{b} - 1 = \frac{c}{d} - 1$

Incorporating -1,  $\frac{a-b}{b} = \frac{c-d}{d}$

Therefore,  $a - b : b :: c - d : d$

Again,  $4 : 2 :: 6 : 3$ , then  $4 - 2 : 2 :: 6 - 3 : 3$

NOTE.—This comparison is sometimes called "*Division*." \*

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\* The technical terms, *Composition* and *Division*, are calculated rather to perplex than to aid the learner, and are properly falling into disuse. The objection to the former is, that it is liable to be mistaken for the *composition* or *compounding* of ratios, whereas the two

## THEOREM IX.

*If two ratios are respectively equal to a third, they are equal to each other.*

Let  $a : b :: m : n$ , and  $c : d :: m : n$

Then  $\frac{a}{b} = \frac{m}{n}$ , and  $\frac{c}{d} = \frac{m}{n}$

By Ax. 1,  $\frac{a}{b} = \frac{c}{d}$ . That is,  $a : b = c : d$

Again,  $12 : 4 = 6 : 2$ , and  $9 : 3 = 6 : 2$   
 $\therefore 12 : 4 = 9 : 3$

## THEOREM X.

*When any number of quantities are proportional, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let  $a : b :: c : d :: e : f$ , etc.

Then  $a : b :: a + c + e : b + d + f$ , etc.

For (Th. 1),  $ad = bc$

And, "  $af = be$

Also,  $ab = ba$

Adding (Ax. 2),  $ab + ad + af = ba + bc + be$

Factoring,  $a(b + d + f) = b(a + c + e)$

Hence, (Th. 3),  $a : b :: (a + c + e, \text{ etc.}) : (b + d + f, \text{ etc.})$

operations are *entirely different*. In one the terms are *added*, in the other they are *multiplied* together. (Art. 351.)

The objection to the latter is, that the change to which the term *division* is here applied, is effected by *subtraction*, and has no reference to *division*, in the sense the word is used in Arithmetic and Algebra. Moreover, the alteration in the terms of Theorem 6 is produced by *actual division*. Usage, however ancient, can *no longer justify* the employment of the *same word* in *two different senses*, in explaining the *same subject*.

### THEOREM XI.

*If the corresponding terms of two or more proportions are multiplied together, the products will be proportional.*

Let  $a : b :: c : d$ , and  $e : f :: g : h$

Then  $ae : bf :: cg : dh$

For,  $\frac{a}{b} = \frac{c}{d}$  and  $\frac{e}{f} = \frac{g}{h}$

Mult. ratios together (Ax. 4),  $\frac{ae}{bf} = \frac{cg}{dh}$

Hence, (Th. 3),  $ae : bf :: cg : dh$ .

### THEOREM XII.

*If four quantities are proportional, like powers or roots of these quantities are proportional.*

Let  $a : b :: c : d$ , then  $\frac{a}{b} = \frac{c}{d}$

By Ax. 10,  $\frac{a^n}{b^n} = \frac{c^n}{d^n}$

Hence (Th. 3),  $a^n : b^n :: c^n : d^n$

Extracting sq. root,  $a^{\frac{1}{2}} : b^{\frac{1}{2}} :: c^{\frac{1}{2}} : d^{\frac{1}{2}}$

Again,  $2 : 3 :: 4 : 6$ , then  $2^2 : 3^2 :: 4^2 : 6^2$

In like manner,  $\sqrt{4} : \sqrt{9} :: \sqrt{16} : \sqrt{36}$ .

NOTE.—The index  $n$  may be either *integral* or *fractional*.

### THEOREM XIII.

*Equimultiples of two quantities are proportional to the quantities themselves.*

Since  $\frac{a}{b} = \frac{c}{d}$ , by Art. 352, Prin. 3,  $\frac{am}{bm} = \frac{c}{d}$

Hence,  $am : bm :: c : d$ .

## PROBLEMS.

1. The first three terms of a proportion are 6, 8, and 3. What is the fourth?

$$\begin{array}{ll} \text{Let} & x = \text{the fourth term,} \\ \text{Then} & 6 : 8 :: 3 : x \\ \therefore 6x = 24, & \text{and } x = 4. \end{array}$$

2. The last three terms of a proportion are 8, 6, and 12. What is the first?

3. Required a third proportional to 25 and 400.

4. Required a mean proportional between 9 and 16.

5. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference to 6.

6. Divide the number 18 into two such parts, that the squares of those parts may be in the ratio of 25 to 16.

7. Divide the number 28 into two such parts, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 32 to 18.

8. What two numbers are those whose product is 24, and the difference of their cubes is to the cube of their difference as 19 to 1?

9. Find two numbers whose sum is to their difference as 9 is to 6, and whose difference is to their product as 1 to 12.

10. A rectangular farm contains 860 acres, and its length is to its breadth as 43 to 32. What are the length and breadth?

11. There are two square fields; a side of one is 10 rods longer than a side of the other, and the areas are as 9 to 4. What is the length of their sides?

12. What two numbers are those whose product is 135, and the difference of their squares is to the square of their difference as 4 to 1?

13. Find two numbers whose product is 320; and the difference of their cubes is to the cube of their difference as 61 to 1.

## CHAPTER XVIII.

### PROGRESSION.

**379.** A *Progression* is a series of quantities which *increase* or *decrease* according to a fixed law.

**380.** The *Terms of a Progression* are the quantities which form the series. The *first* and *last* terms are the *extremes*; the others, the *means*.

**381.** Progressions are of three kinds: *arithmetical*, *geometrical*, and *harmonical*.

### ARITHMETICAL PROGRESSION.

**382.** An *Arithmetical Progression* is a series which *increases* or *decreases* by a *constant quantity* called the *common difference*.

**383.** In an *ascending* series, each term is found by *adding* the *common difference* to the preceding term.

If the first term is  $a$ , and the common difference  $d$ , the series is  
 $a, a+d, a+2d, a+3d$ , etc.

If  $a = 2$ , and  $d = 3$ , the series is 2, 5, 8, 11, 14, etc.

**384.** In a *descending* series, each term is found by *subtracting* the common difference from the preceding term.

If  $a$  is the first term, and  $d$  the common difference, the series is  
 $a, a-d, a-2d, a-3d$ , etc.

In this case, the common difference may be considered  $-d$ . Hence, the common difference may be either *positive* or *negative*. And, since adding a *negative* quantity is equivalent to subtracting an equal

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379. What is a progression? 380. The terms? 381. How many kinds of progression? 382. An arithmetical progression? What is this constant quantity called? 383. An ascending series? 384. A descending series?

*positive* one, it may therefore properly be said that each successive term of the series is derived from the preceding by the *addition* of the *common difference*. (Art. 75, Prin. 3.)

NOTES.—1. The *common difference* was formerly called *arithmetical ratio*; but this term is passing out of use.

2. An Arithmetical Progression is sometimes called an *Equidifferent Series*, or a *Progression by Difference*. In every progression there may be an infinite number of terms.

**385.** *If four quantities are in arithmetical progression, the sum of the extremes is equal to the sum of the means.*

Let  $a, a+d, a+2d, a+3d$ , be the series.

Adding extremes, etc.,  $2a+3d = 2a+3d$ .

Or, let  $2, 2+3, 2+6, 2+9$ , be the series.

Then  $2+2+9 = 2+3+2+6$ .

**386.** *If three quantities are in arithmetical progression, the sum of the extremes is equal to double the mean.*

Let  $a, a+d, a+2d$ , be the series.

Then  $2a+2d = 2(a+d)$ .

Again, let  $2, 2+4, 2+8$ , be the series.

Then  $2+2+8 = 2(2+4)$ .

COR.—An *Arithmetical Mean* between two quantities may be found by taking *half their sum*.

**387.** In *Arithmetical Progression* there are *five elements* to be considered: the *first* term, the *common difference*, the *last* term, the *number* of terms, and the *sum* of the terms.

Let  $a$  = the first term.

$d$  = the common difference,

$l$  = the last term.

$n$  = the number of terms.

$s$  = the sum of the terms.

The relation of these five quantities to each other is such that if any *three* of them are given, the *other two* can be found.

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385. What is true of four quantities in arithmetical progression? 386. Of three quantities? 387. Name the elements in arithmetical progression? What relation have they to each other?

## CASE I.

**388. The First Term, the Common Difference, and Number of Terms being given, to Find the *Last Term*.**

Each succeeding term of a progression is found by adding the common difference to the preceding term. (Art. 384.) Therefore the terms of an *ascending* series are

$$a, a+d, a+2d, a+3d, \text{ etc.}$$

The terms of a *descending* series are

$$a, a-d, a-2d, a-3d, \text{ etc.}$$

It will be seen that the coefficient of  $d$  in each term of both series is *one less* than the number of that term in the series. Therefore, putting  $l$  for the last or  $n$ th term, we have

$$\text{FORMULA I.} \quad l = a \pm (n-1)d.$$

**RULE.—I.** *Multiply the common difference by the number of terms less one.*

**II.** *When the series is ascending, add this product to the first term ; when descending, subtract it from the first term.*

1. Given  $a = 3$ ,  $d = 2$ , and  $n = 7$ , to find  $l$ .

$$l = a \pm (n-1)d = 3 + (7-1)2 = 15, \text{ Ans.}$$

2. Given  $a = 25$ ,  $d = -2$ , and  $n = 9$ , to find  $l$ .

3. Given  $a = 12$ ,  $d = 4$ , and  $n = 15$ , to find  $l$ .

4. Given  $a = 1$ ,  $d = -\frac{1}{2}$ , and  $n = 13$ , to find  $l$ .

5. Given  $a = \frac{3}{4}$ ,  $d = \frac{1}{8}$ , and  $n = 9$ , to find  $l$ .

6. Given  $a = 1$ ,  $d = -.01$ , and  $n = 10$ , to find  $l$ .

7. Find the 12th term of the series 3, 5, 7, 9, 11, etc.

**NOTE.**—In this problem,  $a = 3$ ,  $d = 2$ ,  $n = 12$ . *Ans.* 25.

8. Find the 15th term of 1, 4, 7, 10, etc.

9. Find the 9th term of 31, 29, 27, 25, etc.

10. What is the 30th term of the series 1,  $2\frac{1}{2}$ , 4,  $5\frac{1}{2}$ , etc.

11. Find the 25th term of the series  $x+3x+5x+7x$ , etc.

12. Find the  $n$ th term of the series 2a, 5a, 8a, 11a, etc.

## CASE II.

**389. The Extremes and Number of Terms being given, to Find the Sum of the Series.**

Let  $a, a+d, a+2d, a+3d \dots l$ , be an arithmetical progression, the sum of which is required.

Since the sum of two or more quantities is the same in whatever order they are added (Art. 63, Prin. 2), we have

$$\begin{array}{rcl} & s = a + (a+d) + (a+2d) + (a+3d) + \dots + l \\ \text{Inverting,} & s = l + (l-d) + (l-2d) + (l-3d) + \dots + a \\ \text{Adding,} & 2s = a + l + (a+l) + (a+l) + (a+l) + \dots + a+l \end{array}$$

$\therefore 2s = (a+l)$  taken  $n$  times, or as many times as there are terms in the series.

That is,  $2s = (a+l)n$ . Hence, the

$$\text{FORMULA II.} \quad s = \frac{(a+l)}{2} \times n.$$

**RULE.**—Multiply half the sum of the extremes by the number of terms.

**COR.**—From the preceding illustration it follows that the sum of the extremes is equal to the sum of any two terms equally distant from the extremes.

Thus, in the series, 3, 5, 7, 9, 11, 13, the sum of the first and last terms, of the second and fifth, etc., is the same, viz., 16.

1. Given  $a = 4$ ,  $l = 148$ , and  $n = 15$ , to find  $s$ .

**SOLUTION.**  $4 + 148 = 152$ , and  $(152 \div 2) \times 15 = 1140$ , *Ans.*

2. Given  $a = \frac{1}{2}$ ,  $l = 30$ , and  $n = 50$ , to find  $s$ .

3. Given  $a = 6$ ,  $l = 42$ , and  $n = 9$ , to find  $s$ .

4. Given  $a = 5$ ,  $l = 75$ , and  $n = 35$ , to find  $s$ .

5. Given  $a = 2$ ,  $l = 1$ , and  $n = 17$ , to find  $s$ .

6. Find the sum of the series 2, 5, 8, 11, etc., to 20 terms.

7. Find the sum of the series 1,  $1\frac{1}{2}$ , 2,  $2\frac{1}{2}$ , etc., to 25 terms.

8. Find the sum of the series 75, 72, 69, 66, 63, etc., to 15 terms.

**390.** The two preceding formulas are fundamental, and furnish the means for solving all the problems in Arithmetical Progression. From them may be derived *eighteen other formulas*.

### BY FORMULA I.

**391.** This formula contains *four different* quantities; the *first* term, the *common difference*, the *last* term, and the *number* of terms. If *any three* of these quantities are given, the *other* may be found. (Art. 388.)

1.  $l = a \pm (n - 1)d$ ;  $a, d$ , and  $n$  being given.

3. Given  $d, l$ , and  $n$ , to find  $a$ , the first term.\*

Transposing  $(n-1)d$  in (1),

$$a = l \pm (n-1)d.$$

4. Given  $a, l$ , and  $n$ , to find  $d$ , the common difference.

Transposing in (1), and dividing by  $(n-1)$ ,

$$d = \frac{l - a}{n - 1}.$$

5. Given  $a, d$ , and  $l$ , to find  $n$ , the number of terms.

Clearing of fractions and reducing (4),

$$n = \frac{l - a}{d} + 1.$$

1. Given  $a = 25, d = 3$ , and  $n = 12$ , to find  $l$ .

2. Given  $a = 58, d = 5$ , and  $n = 45$ , to find  $l$ .

3. Given  $d = 3, l = 35$ , and  $n = 9$ , to find  $a$ .

4. Given  $l = 57, d = 5$ , and  $n = 21$ , to find  $a$ .

5. Given  $a = 15, l = 85$ , and  $n = 31$ , to find  $d$ .

6. Given  $a = 28, l = 7$ , and  $n = 26$ , to find  $d$ .

7. Given  $a = 23, d = 5$ , and  $l = 5138$ , to find  $n$ .

8. Given  $a = 6, d = 6$ , and  $l = 1152$ , to find  $n$ .

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\* For Formula 2, see Art. 389.

## BY FORMULA II.

**392.** In this formula there are *four different* quantities: the *first* term, the *last* term, the *number* of terms, and the *sum* of the terms. If *any three* of these quantities are given, the other may be found. (Art. 389.)

$$2. \quad s = \frac{a + l}{2} \times n, \quad a, l, \text{ and } n \text{ being given.}$$

NOTE.—For Formulas 3–5, see Article 391.

6. Given  $l$ ,  $n$ , and  $s$ , to find  $a$ , the first term.

Clearing (2) of fractions, dividing and transposing,

$$a = \frac{2s}{n} - l.$$

7. Given  $a$ ,  $n$ , and  $s$ , to find  $l$ , the last term.

Transposing in (6), we have

$$l = \frac{2s}{n} - a.$$

8. Given  $a$ ,  $l$ , and  $s$ , to find  $n$ , the number of terms.

Clearing (7) of fractions, transposing, factoring, and dividing,

$$n = \frac{2s}{a + l}.$$

1. Given  $a = 9$ ,  $l = 41$ , and  $n = 7$ , to find  $s$ .
2. Given  $a = \frac{1}{2}$ ,  $l = 45$ , and  $n = 50$ , to find  $s$ .
3. Given  $l = 50$ ,  $d = 4$ , and  $n = 12$ , to find  $a$ .
4. Given  $a = 9$ ,  $l = 41$ , and  $s = 150$ , to find  $n$ .
5. Given  $d = 7$ ,  $l = 21$ , and  $n = 35$ , to find  $a$ .
6. Given  $a = 46$ ,  $l = 24$ , and  $s = 455$ , to find  $n$ .
7. Given  $a = 27$ ,  $n = 9$ , and  $s = 72$ , to find  $l$ .
8. Given  $a = 72$ ,  $n = 8$ , and  $s = 288$ , to find  $l$ .
9. Find the sum of the series 3, 5, 7, 9, etc., to 15 terms.
10. Find the twentieth term of 5, 8, 11, 14, 17, etc.
11. If the first term of an ascending series is 5, and the common difference 4, what is the 15th term?

**393.** The remaining twelve formulas are derived by combining the preceding ones in such a manner as to eliminate the quantity whose value is not sought. They are contained in the following

TABLE.

No.	GIVEN.	REQUIRED.	FORMULAS.
9.	$d, n, s$	$a$	$a = \frac{2s - dn^2 + dn}{2n}$
10.	$d, l, s$	$a$	$a = \frac{d}{2} \pm \sqrt{\left(l + \frac{d}{2}\right)^2 - 2ds}$
11.	$a, l, s$	$d$	$d = \frac{l^2 - a^2}{2s - l - a}$
12.	$l, n, s$	$d$	$d = \frac{2(nl - s)}{n(n - 1)}$
13.	$a, n, s$	$d$	$d = \frac{2s - 2an}{n^2 - n}$
14.	$d, n, s$	$l$	$l = \frac{s}{n} + \frac{(n - 1)d}{2}$
15.	$a, d, s$	$l$	$l = -\frac{d}{2} \pm \sqrt{2ds + \left(a - \frac{d}{2}\right)^2}$
16.	$a, d, s$	$n$	$n = \frac{\pm \sqrt{(2a - d)^2 + 8ds} - 2a + d}{2d}$
17.	$d, l, s$	$n$	$n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}$
18.	$a, d, n$	$s$	$s = \frac{n}{2} [2a + (n - 1)d]$
19.	$a, d, l$	$s$	$s = \frac{l + a}{2} + \frac{l^2 - a^2}{2d}$
20.	$d, l, n$	$s$	$s = \frac{n}{2} [2l - (n - 1)d]$

**[39]** Of the twenty formulas in Arithmetical Progression, the *first two* are *indispensable*, and should be *thoroughly committed* to memory; the *next six* are important in the solution of particular problems. The remaining twelve are of less consequence, but will be found interesting to the inquisitive student.

**394.** By the fourth formula in Art. 391, any number of arithmetical means may be inserted between two given terms of an arithmetical progression. For, the *number* of terms consists of the *two extremes* and all the *intermediate* terms.

Let  $m$  = the number of means to be inserted.

Then  $m + 2 = n$ , the whole number of terms.

Substituting  $m + 2$  for  $n$  in the fourth formula, we have

$$d = \frac{l-a}{m+1}. \text{ Hence,}$$

*The required number of means is found by the continued addition of the common difference to the successive terms.*

1. Find 4 arithmetical means between 1 and 31.
2. Find 9 arithmetical means between 3 and 48.

### PROBLEMS.

1. If the first term of an ascending series is 5, the common difference 3, and the number of terms 15, what is the last term?

2. If the first term of a descending series is 27, the common difference 3, and the number of terms 12, what is the last term?

3. If the first term of an ascending series be 7, and the common difference 5, what will the 20th term be?

4. Find 5 arithmetical means between 2 and 60.

5. What is the sum of 100 terms of the series  $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3$ , etc.

6. If the sum of an arithmetical series is 18750, the <sup>last</sup> least term 5, and the number of terms 20, what is the common difference?

7. Required the sum of the odd numbers 1, 3, 5, 7, 9, 11, etc., continued to 76 terms?

8. Required the sum of 100 terms of the series of even numbers 2, 4, 6, 8, 10, etc.

9. The extremes of a series are 2 and 47, and the number of terms is 10. What is the common difference?

10. Insert 8 means between 6 and 72.

11. Insert 9 means between 12 and 108.

12. The first term of a descending series is 100, the common difference 5, and the number of terms 15. What is the sum of the terms?

NOTE.—1. In Arithmetical Progression, problems often occur in which the terms are not directly given, but are implied in the conditions. Such problems may be solved by stating the conditions algebraically, and reducing the equations.

13. Find four numbers in arithmetical progression, whose sum shall be 48, and the sum of their squares 656.

Let  $x$  = the second of the four numbers.

And  $y$  = their common difference.

By the conditions,  $(x-y) + x + (x+y) + (x+2y) = 48$  (1)

And  $(x-y)^2 + x^2 + (x+y)^2 + (x+2y)^2 = 656$  (2)

Uniting terms in (1),  $4x + 2y = 48$  (3)

“ “ “ (2),  $4x^2 + 4xy + 6y^2 = 656$  (4)

Transposing and dividing in (3),  $y = 24 - 2x$  (5)

Dividing (4) by 2,  $2x^2 + 2xy + 3y^2 = 328$  (6)

Substituting value of  $y$ ,  $2x^2 + 2x(24 - 2x) + 3(24 - 2x)^2 = 328$

Reducing,  $x^2 - 24x = -140$

Completing square, etc.,  $x = 14$  or  $10$

Substituting in (5)  $y = -4$  or  $4$

Hence the required numbers are 6, 10, 14, and 18.

NOTE.—2. The first two values of  $x$  and  $y$  produce a *descending* series; the other two an *ascending* series. In both the numbers are the same.

14. Find three numbers in arithmetical progression whose sum is 15, and the sum of their cubes is 495.

15. If 100 marbles are placed in a straight line a yard apart, how far must a person travel to bring them one by one to a box a yard from the first marble?

16. How many strokes does a common clock strike in 24 hours?

17. A student bought 25 books, and gave 10 cents for the first, 30 cents for the second, 50 cents for the third, etc. What did he pay for the whole?

18. A boy puts into his bank a cent the first day of the year, 2 cents the second day, 3 cents the third day, and so on to the end of the year. What sum does he thus lay up in 365 days?

19. The clocks of Venice go on to 24 o'clock. How many strokes does one of them strike in a day?

20. What will be the amount of \$1, at 6 per cent simple interest, in 20 years?

21. What three numbers are those whose sum is 120, and the sum of whose squares is 5600?

22. A traveller goes 10 miles a day; three days after, another follows him, who goes 4 miles the first day, 5 the second, 6 the third, and so on. When will he overtake the first?

23. Find four numbers, such that the sum of the squares of the extremes is 4500, and the sum of the squares of the means is 4100.

24. A sets out from a certain place and goes 1 mile the first day, 3 miles the second day, 5 the third, etc. After he has been gone 3 days, he is followed by B, who goes 11 miles the first day, 12 the second, etc. When will B overtake A?

25. The first term of a decreasing arithmetical progression is 10, the common difference  $\frac{1}{3}$ , and the number of terms 21. Required the sum of the series.

26. A debt can be discharged in 60 days by paying \$1 the first day, \$4 the second, \$7 the third, etc. Required the amount of the debt and of the last payment.

## GEOMETRICAL PROGRESSION.

**395.** A *Geometrical Progression* is a series of quantities which *increase* or *decrease* by a *constant multiplier* called the *ratio*. Hence,

The *ratio* may be an *integer* or a *fraction*.

NOTE,—When the ratio is *fractional*, the series will decrease. For multiplying by a fraction is taking a certain part of the multiplicand as many times as there are like parts of a unit in the multiplier.

**396.** In a geometrical series, each succeeding term is found by multiplying the preceding one by the ratio.

Thus, if  $a$  is the first term, and  $r$  the ratio, the series is

$$a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, \text{ etc.}$$

If the ratio is 3, the series is

$$a, a \times 3, a \times 3^2, a \times 3^3, \text{ etc.}$$

If the ratio is  $\frac{1}{3}$ , the series is

$$a, a \times \frac{1}{3}, a \times \frac{1}{3} \times \frac{1}{3}, a \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}, \text{ etc.}$$

**397.** An *Ascending Series* is one which increases by an integral ratio; as, 2, 4, 8, 16, 32, etc.

**398.** A *Descending Series* is one which decreases by a fractional ratio; as, 64, 32, 16, 8, etc.

**399.** When the ratio is a *positive* quantity, all the terms of the progression are *positive*; when it is *negative*, the terms are alternately *positive* and *negative*.

Thus, if the first term is  $a$ , and the ratio  $-3$ , the series is

$$a, -3a, +9a, -27a, +81a, \text{ etc.}$$

395. What is a geometrical progression? 397. What is an ascending series?  
398. Descending? 399. What law governs the signs?

**400.** In geometrical progression there are five elements: the *first* term, the *last* term, the *number* of terms, the *common ratio*, and the *sum* of the terms.

Let  $a$  = the first term,  
 $l$  = the last term,  
 $n$  = the number of terms,  
 $r$  = the ratio,  
 $s$  = the sum of the terms.

The relation of these five quantities to each other is such that if *any three* of them are given, the *other two* can be found.

### CASE I.

**401.** The First Term, the Number of Terms, and the Ratio being given, to Find the Last Term.

In this problem,  $a$ ,  $n$ , and  $r$  are given, to find  $l$ , the last term.

The successive terms of the series are

$a$ ,  $ar$ ,  $ar^2$ ,  $ar^3$ ,  $ar^4$ , etc., to  $ar^{n-1}$ . (Art. 397.)

By inspection, it will be seen that the ratio  $r$  consists of a regular series of powers, and in each term the index of the power is *one less* than the number of the terms. Therefore, the last or  $n$ th term of the series is  $ar^{n-1}$ . Hence, we have

FORMULA I.  $l = ar^{n-1}$ .

**RULE.**—*Multiply the first term by that power of the ratio whose index is one less than the number of terms.*

**COR.**—Any term in a series may be found by the preceding rule; for the series may be supposed to stop at that term.

1. Given  $a = 5$ ,  $n = 6$ , and  $r = 2$ , to find  $l$ .
2. Given  $a = 2$ ,  $n = 8$ , and  $r = 3$ , to find  $l$ .
3. Given  $a = 72$ ,  $n = 5$ , and  $r = \frac{1}{2}$ , to find  $l$ .
4. Given  $a = 5$ ,  $n = 4$ , and  $r = 4$ , to find  $l$ .
5. Given  $a = 7$ ,  $n = 5$ , and  $r = 2$ , to find  $l$ .
6. Given  $a = 10$ ,  $n = 6$ , and  $r = -5$ , to find  $l$ .

## CASE II.

**402.** The First Term, the Last Term, and the Ratio being given, to Find the *Sum* of the Terms.

In this problem,  $a$ ,  $l$ , and  $r$  are given, to find  $s$ .

Since  $s$  = the sum of the terms, we have

$$s = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying (1) by  $r$ ,

$$rs = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n. \quad (2)$$

$$\text{Subtracting (1) from (2),} \quad rs - s = ar^n - a. \quad (3)$$


$$\text{Factoring and dividing,} \quad s = \frac{ar^n - a}{r - 1} \quad (4)$$

In equation (4),  $ar^n$  is the last term of (2), and is therefore the product of the ratio by the last term in the given series.

Substituting  $lr$  for  $ar^n$ , we have

$$\text{FORMULA II.} \quad s = \frac{lr - a}{r - 1}.$$

**RULE.**—*Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.*

 For the method of finding the sum of an *infinite descending series*, see Art. 435.

1. Given  $a = 2$ ,  $l = 500$ , and  $r = 3$ , to find the sum.

$$\text{SOLUTION.} \quad s = \frac{lr - a}{r - 1} = \frac{500 \times 3 - 2}{2} = 749, \text{ Ans.}$$

2. Given  $a = 3$ ,  $l = 9375$ , and  $r = 5$ , to find  $s$ .

3. Given  $a = 9$ ,  $l = 9000$ , and  $r = 10$ , to find  $s$ .

4. Given  $a = 5$ ,  $l = 20480$ , and  $r = 4$ , to find  $s$ .

5. Given  $a = 15$ ,  $l = 3240$ , and  $r = 6$ , to find  $s$ .

6. Given  $a = 25$ ,  $l = 6400$ , and  $r = 4$ , to find  $s$ .

402. How find the sum of the terms ?

**403.** The two preceding formulas furnish the means for solving all problems in geometrical progression. They may be varied so as to form *eighteen* other formulas.

### BY FORMULA I.

**404.** The first formula contains *four different* quantities: the *first* term, the *last* term, the *ratio*, and the *number* of terms. If *any three* of these quantities are given, the *other* may be found. By the first formula,

1.  $l = ar^{n-1}$ ;  $a$ ,  $n$ , and  $r$  being given. (Art. 401.)

For formula 2, see Article 402.

3. Given  $l$ ,  $n$ , and  $r$ , to find  $a$ , the first term.

Factoring (1), and dividing by  $r^{n-1}$ ,

$$a = \frac{l}{r^{n-1}}.$$

4. Given  $a$ ,  $l$ , and  $n$ , to find  $r$ , the ratio.

Dividing (1) by  $a$ , and extracting the root denoted by the index,

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

5. Given  $a$ ,  $l$ , and  $r$ , to find  $n$ , the number of terms.

Dividing (1) by  $a$ ,  $r^{n-1} = \frac{l}{a}$ .

By logarithms,  $\log r(n-1) = \log l - \log a$

Dividing, etc.,  $n = \frac{\log l - \log a}{\log r} + 1.$

NOTE.—Since this formula contains logarithms, it may be deferred till that subject is explained.

1. Given  $a = 3$ ,  $n = 5$ , and  $r = 10$ , to find  $l$

2. Given  $a = 5$ ,  $n = 6$ , and  $r = 5$ , to find  $l$

3. Given  $l = 256$ ,  $n = 8$ , and  $r = 2$ , to find  $a$ .

4. Given  $l = 243$ ,  $n = 5$ , and  $r = 3$ , to find  $a$ .

5. Given  $a = 2$ ,  $l = 2592$ , and  $n = 5$ , to find  $r$ .

6. Given  $a = 4$ ,  $l = 2500$ , and  $n = 5$ , to find  $r$ .

## BY FORMULA II.

**405.** This formula contains four different quantities: the *first* term, the *last* term, the *ratio*, and the *sum* of the terms. If *any three* of them are given, the *other* may be found. By the second formula,

$$2. \quad s = \frac{lr - a}{r - 1}, \quad a, l, \text{ and } r \text{ being given.} \quad (\text{Art. 402.})$$

For formulas 3-5, see Article 404.

6. Given  $l$ ,  $r$ , and  $s$ , to find  $a$ , the first term.

Clearing (2) of fractions, etc.,

$$a = lr - s(r - 1)$$

7. Given  $a$ ,  $r$ , and  $s$ , to find  $l$ , the last term.

Transposing in (6),

$$lr = a + s(r - 1).$$

Dividing by  $r$ ,

$$l = \frac{a + s(r - 1)}{r}.$$

8. Given  $a$ ,  $l$ , and  $s$ , to find  $r$ , the ratio.

Clearing (2) of fractions,

$$sr - s = lr - a.$$

Transposing in the last equation,

$$sr - lr = s - a.$$

Factoring, etc.,

$$r = \frac{s - a}{s - l}.$$

1. Given  $a = 2$ ,  $l = 162$ , and  $r = 3$ , to find  $s$ .
2. Given  $l = 54$ ,  $r = 3$ , and  $s = 80$ , to find  $a$ .
3. Given  $a = 4$ ,  $r = 5$ , and  $s = 624$ , to find  $l$ .
4. Given  $a = 4$ ,  $l = 12500$ , and  $s = 15624$ , to find  $r$ .
5. Given  $a = 5$ ,  $l = 180$ , and  $r = 6$ , to find  $s$ .
6. Given  $a = 7$ ,  $r = 3$ , and  $s = 847$ , to find  $l$ .

406. The remaining twelve formulas are derived by combining the preceding ones in such a manner as to eliminate the quantity whose value is not sought.

TABLE.

No.	GIVEN.	REQUIRED.	FORMULAS.
9.	$n, r, s$	$a$	$a = \frac{(r-1)s}{r^n - 1}.$
10.	$l, n, s$	$a$	$a(s-a)^{n-1} = l(s-l)^{n-1}.$
11.	$a, n, s$	$l$	$l(s-l)^{n-1} = a(s-a)^{n-1}.$
12.	$n, r, s$	$l$	$l = \frac{(r-1)sr^{n-1}}{r^n - 1}.$
13.	$a, l, s$	$n$	$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1$
14.	$a, r, s$	$n$	$n = \frac{\log[a + (r-1)s] - \log a}{\log r}.$
15.	$l, r, s$	$n$	$n = \frac{\log l - \log[lr - (r-1)s]}{\log r} + 1$
16.	$a, n, s$	$r$	$r^n - \frac{s}{a}r = 1 - \frac{s}{a}.$
17.	$l, n, s$	$r$	$r^n + \frac{s}{l-s}r^{n-1} = \frac{s}{l-s}.$
18.	$a, n, r$	$s$	$s = \frac{a(r^n - 1)}{r - 1}.$
19.	$l, n, r$	$s$	$s = \frac{lr^n - l}{r^n - r^{n-1}}.$
20.	$a, l, n$	$s$	$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$

Of the twenty formulas in Geometrical Progression, the *first two* are *fundamental*, and should be *thoroughly committed* to memory; the *next six* are important in the solution of particular problems. The remainder are less practical.

**407.** By the fourth formula (Art. 404), any number of geometrical means may be found between two given quantities.

Let  $m$  = the number of means required.

Then  $m + 2 = n$ .

Substituting  $m + 2$  for  $n$  in the formula, we have

$$r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}.$$

The ratio being found, the means required are obtained by continued multiplication.

1. Find two geometrical means between 3 and 192.

$$\text{SOLUTION. } r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} = \sqrt[3]{\frac{192}{3}} = \sqrt[3]{64} = 4.$$

The ratio being 4, the first mean is  $3 \times 4 = 12$ ; the second is

$$12 \times 4 = 48.$$

2. Find three geometrical means between  $\frac{1}{2}$  and 128.

### PROBLEMS.

1. In a geometrical progression, the first term is 6, the last term 2916, and the ratio 3. What is the sum of all the terms?

2. In a decreasing geometrical series, the first term is  $\frac{1}{2}$ , the ratio  $\frac{1}{3}$ , and the number of terms 8. What is the sum of the series?

3. What is the sum of the series 1, 3, 9, 27, etc., to 15 terms?

4. Find the sum of 12 terms of the series, 1,  $\frac{2}{3}$ ,  $\frac{4}{9}$ ,  $\frac{8}{27}$ , etc.

5. If the first term of a series is 2, the ratio 3 and the number of terms 15, what is the last term?

6. What is the 16th term of a series, the first term of which is 3, and the ratio 3?

NOTE.—When the terms of the series are not stated directly, they may be represented algebraically.

7. Find three numbers in geometrical progression, such that their sum shall be 21, and the sum of their squares 189.

Let the three numbers be  $x$ ,  $\sqrt{xy}$ , and  $y$ .

By the conditions,  $x + \sqrt{xy} + y = 21$  (1)

And  $x^2 + xy + y^2 = 189$  (2)

Transposing and sq. (1),  $x^2 + 2xy + y^2 = 441 - 42\sqrt{xy} + xy$  (3)

Subtracting (2) from (3),  $xy = 252 - 42\sqrt{xy} + xy$  (4)

Transposing, etc.,  $\sqrt{xy} = 6$  (5)

Involving,  $xy = 36$  (6)

And  $3xy = 108$  (7)

Subtracting (7) from (2),  $x^2 - 2xy + y^2 = 81$

Extracting root,  $x - y = 9$  (8)

Substituting (5) in (1),  $x + y = 15$  (9)

Combining (8) and (9),  $x = 12$

$y = 3$

Hence the numbers, 12, 6, and 3, *Ans.*

8. A father gives his daughter \$1 on New Year's day towards her portion, and doubles it on the first day of every month through the year. What is her portion?

9. A dairyman bought 10 cows, on the condition that he should pay 1 cent for the first, 3 for the second, 9 for the third, and so on to the last. What did he pay for the last cow and for the ten cows?

10. A man buys an umbrella, giving 1 cent for the first brace, 2 cents for the second brace, 4 for the third, and so on, there being 10 braces. What is the cost of the umbrella?

11. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. Find the numbers.

12. What would be the price of a horse, if he were to be sold for the 32 nails in his shoes, paying 1 mill for the first, 2 mills for the second, 4 for the third, and so on?

13. Find four numbers in geometrical progression, such that the sum of the first three is 130, and that of the last three is 390.

14. A man divides \$210 in geometrical progression among three persons; the first had \$90 more than the last. How much did each receive?

15. There are five numbers in geometrical progression. The sum of the first four is 468, and that of the last four is 2340. What are the numbers?

16. The sum of \$700 is divided among 4 persons, whose shares are in geometrical progression; and the difference between the extremes is to the difference between the means as 37 to 12. What are the respective shares?

17. The population of a town increases annually in geometrical progression, rising in four years from 10000 to 14641. What is the ratio of annual increase?

18. The sum of four numbers in geometrical progression is 15, and the sum of their squares 85. What are the numbers?

## HARMONICAL PROGRESSION.\*

**408.** An *Harmonical Progression* is such, that of any three *consecutive* terms, the first is to the third as the difference of the first and second is to the difference of the second and third.

Thus, 10, 12, 15, 20, 30, 60,  
are in harmonic progression; for

$$10 : 15 :: 12 - 10 : 15 - 12$$

$$12 : 20 :: 15 - 12 : 20 - 15$$

$$15 : 30 :: 20 - 15 : 30 - 20$$

$$20 : 60 :: 30 - 20 : 60 - 30$$

Let  $a, b, c, d, e, f, g$ , be an Harmonical progression, then

$$a : c : a - b : b - c, \text{ etc.}$$

NOTE.—When three quantities are such, that the *first* is to the *third* as the *difference* of the first and second is to the *difference* of the second and third, they are said to be in *Harmonical Proportion*.

Thus, 2, 3, and 6, are in harmonical proportion.

408. What is an harmonic progression?

\* If a musical string be divided in harmonical proportion, the different parts will vibrate in harmony. Hence, the name.

**409. To Find the *Third Term* of an Harmonical Progression, the First Two being given.**

Let  $a$  and  $b$  be the first two terms, and  $x$  the third term.

Then  $a : x :: a-b : b-x$

Multiplying extremes, etc.,  $ab-ax = ax-bx$

Transposing, etc.,  $2ax-bx = ab$

Factoring, and dividing by  $2a-b$ , we have the

$$\text{FORMULA. } x = \frac{ab}{2a-b}.$$

**RULE.**—*Divide the product of the first two terms by twice the first minus the second term; the quotient will be the third term.*

**NOTE.**—This rule furnishes the means for extending an harmonic progression, by adding one term at a time to the two preceding terms.

1. Find the third term in the harmonic series of which 12 and 8 are the first two terms. *Ans.* 6.

2. Find the third term in the harmonic series of which 12 and 18 are the first two terms. *Ans.* 36.

3. If the first two terms of an harmonic progression are 15 and 20, what is the third term? *Ans.* 30.

4. Continue the series 12, 15, 20, for two terms. *Ans.* 30 and 60.

5. Continue the series  $7\frac{1}{2}$ , 9, 12, for two terms. *Ans.* 18 and 36.

**410. To Find a *Mean* or Middle Term between Two Terms of an Harmonic Progression.**

Let  $a$  and  $c$  be the first and third of three consecutive terms of an harmonic progression, and  $m$  the mean.

Then  $a : c :: a-m : m-c$

Mult. extremes and means,  $am-ac = ac-cm$

Transposing and uniting,  $am+cm = 2ac$

Factoring and dividing by  $a+c$ , we have the

$$\text{FORMULA. } m = \frac{2ac}{a+c}.$$

**RULE.**—Divide twice the product of the first and third terms by their sum; the quotient will be the mean or middle term.

6. The first and third of three consecutive terms of an harmonic progression are 9 and 18. Required the mean or middle term.

**SOLUTION.**  $2 \times 9 \times 18 = 324$ , and  $9 + 18 = 27$ .

Now  $324 \div 27 = 12$ , *Ans.*

7. Find an harmonic mean between 12 and 20. *Ans.* 15.

8. Find an harmonic mean between 15 and 30. *Ans.* 20.

**411.** The *Reciprocals* of the terms of an harmonic progression form an *arithmetical* progression.

Thus, the reciprocals of 10, 12, 15, 20, etc., viz.,

$$\frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{20}, \frac{1}{30}, \text{ etc.,}$$

are an arithmetical progression, whose common difference is  $\frac{1}{60}$ .

Again, let  $a, b, c$  be in harmonic progression.

Then  $a : c :: a - b : b - c$

Mult. extremes and means,  $ab - ac = ac - bc$

Dividing by  $abc$ ,  $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$ . (Art. 324.)

Conversely, the reciprocals of an arithmetical progression form an harmonic progression. Thus,

The reciprocals of the arithmetical progression 1, 2, 3, 4, 5, etc., viz.,  $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ , etc., are in harmonic progression.

**412.** If the *lengths* of six musical strings of equal weight and *tension*, are in the *ratio* of the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \text{ etc.,}$$

the second will sound an *octave* above the first; the third will sound the *twelfth*; the fourth the *double octave*, etc.

410. How find a mean between two terms of an harmonic progression?  
 411. What do the reciprocals of an harmonical progression form?

**409. To Find the *Third Term* of an Harmonical Progression, the First Two being given.**

Let  $a$  and  $b$  be the first two terms, and  $x$  the third term.

Then  $a : x :: a-b : b-x$

Multiplying extremes, etc.,  $ab - ax = ax - bx$

Transposing, etc.,  $2ax - bx = ab$

Factoring, and dividing by  $2a - b$ , we have the

$$\text{FORMULA. } x = \frac{ab}{2a - b}.$$

**RULE.**—*Divide the product of the first two terms by twice the first minus the second term; the quotient will be the third term.*

**NOTE.**—This rule furnishes the means for extending an harmonic progression, by adding one term at a time to the two preceding terms.

1. Find the third term in the harmonic series of which 12 and 8 are the first two terms. *Ans.* 6.

2. Find the third term in the harmonic series of which 12 and 18 are the first two terms. *Ans.* 36.

3. If the first two terms of an harmonic progression are 15 and 20, what is the third term? *Ans.* 30.

4. Continue the series 12, 15, 20, for two terms. *Ans.* 30 and 60.

5. Continue the series  $7\frac{1}{2}$ , 9, 12, for two terms. *Ans.* 18 and 36.

**410. To Find a *Mean* or *Middle Term* between Two Terms of an Harmonic Progression.**

Let  $a$  and  $c$  be the first and third of three consecutive terms of an harmonic progression, and  $m$  the mean.

Then  $a : c :: a-m : m-c$

Mult. extremes and means,  $am - ac = ac - cm$

Transposing and uniting,  $am + cm = 2ac$

Factoring and dividing by  $a + c$ , we have the

$$\text{FORMULA. } m = \frac{2ac}{a + c}.$$

**RULE.**—*Divide twice the product of the first and third terms by their sum ; the quotient will be the mean or middle term.*

6. The first and third of three consecutive terms of an harmonic progression are 9 and 18. Required the mean or middle term.

**SOLUTION.**  $2 \times 9 \times 18 = 324$ , and  $9 + 18 = 27$ .

Now  $324 \div 27 = 12$ , *Ans.*

7. Find an harmonic mean between 12 and 20. *Ans.* 15.

8. Find an harmonic mean between 15 and 30. *Ans.* 20.

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Thus, the reciprocals of 10, 12, 15, 20, etc., viz.,

$$\frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{20}, \frac{1}{30}, \text{ etc.},$$

are an arithmetical progression, whose common difference is  $\frac{1}{60}$ .

Again, let  $a, b, c$  be in harmonic progression.

$$\text{Then} \quad a : c :: a - b : b - c$$

$$\text{Mult. extremes and means,} \quad ab - ac = ac - bc$$

$$\text{Dividing by } abc, \quad \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a} \quad (\text{Art. 324.})$$

Conversely, the reciprocals of an arithmetical progression form an harmonic progression. Thus,

The reciprocals of the arithmetical progression 1, 2, 3, 4, 5, etc., viz.,  $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ , etc., are in harmonic progression.

**412.** If the *lengths* of six musical strings of equal weight and *tension*, are in the ratio of the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \text{ etc.},$$

the second will sound an *octave* above the first; the third will sound the *twelfth*; the fourth the *double octave*, etc.

410. How find a mean between two terms of an harmonic progression?

411. What do the reciprocals of an harmonical progression form?

**418. To Expand a Compound Surd into an Infinite Series**6. Reduce  $\sqrt{1+x}$  to an infinite series.

OPERATION.

$$\begin{array}{r}
 1+x \left| 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}, \text{ etc.}, \text{ Ans.} \\
 \underline{2 + \frac{x}{2} \left| + x \right.} \\
 \phantom{2 + \frac{x}{2} \left| + x \right.} + x + \frac{x^2}{4} \\
 \underline{2 + x - \frac{x^2}{8} \left| - \frac{x^2}{4} \right.} \\
 \phantom{2 + x - \frac{x^2}{8} \left| - \frac{x^2}{4} \right.} - \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
 \underline{2 + x - \frac{x^2}{4} + \frac{x^3}{16} \left| + \frac{x^3}{8} - \frac{x^4}{64} \right.}, \text{ etc.} \quad \text{Hence, the}
 \end{array}$$

**RULE.**—*Extract the square root of the given surd.*  
(Art. 298.)

7. Expand  $\sqrt{x^2-y^2}$ . *Ans.*  $x - \frac{y^2}{2x} - \frac{y^4}{8x^3} - \frac{y^6}{16x^5}, \text{ etc.}$

8. Expand  $\sqrt{2}$ , or  $\sqrt{1+1}$ . *Ans.*  $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16}, \text{ etc.}$

**419. The Binomial Theorem applied to the Formation of Infinite Series.**

The Binomial Theorem may often be employed with advantage, in finding the roots of binomials. For a root is expressed like a power, except the exponent of one is an *integer*, and that of the other is a *fraction*.

9. Expand  $(x+y)^{\frac{1}{2}}$  into an infinite series.

**SOLUTION.**—The terms without coefficients are

$$x^{\frac{1}{2}}, x^{-\frac{1}{2}}y, x^{-\frac{3}{2}}y^2, x^{-\frac{5}{2}}y^3, x^{-\frac{7}{2}}y^4, \text{ etc.}$$

The coefficient of the second term is  $+\frac{1}{2}$ ; of the 3d,  $\frac{\frac{1}{2} \times -\frac{1}{2}}{2} = -\frac{1}{8}$ ;

of the 4th term,  $\frac{-\frac{1}{2} \times -\frac{3}{2}}{3} = +\frac{1}{16}, \text{ etc.}$

The series is  $x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y - \frac{1}{8}x^{-\frac{3}{2}}y^2 + \frac{1}{16}x^{-\frac{5}{2}}y^3, \text{ etc.}$

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418. How expand a surd into an infinite series?

**420.** When the *index* of the required power of a binomial is a *positive integer*, the series will *terminate*. For, the index of the leading quantity continually decreases by 1; and soon becomes 0; then the series must stop. (Art. 269.)

**421.** When the index of the required power is *negative*, the series will never terminate. For, by the successive subtractions of a *unit* from the index, it will never become 0; and the series may be continued indefinitely.

**10.** Expand  $(x^2 + y)^{\frac{1}{2}}$  into an infinite series, keeping the factors of the coefficients distinct.

$$(x^2 + y)^{\frac{1}{2}} = x + \frac{y}{2x} - \frac{y^2}{2 \cdot 4x^3} + \frac{3y^3}{2 \cdot 4 \cdot 6x^5} - \frac{3 \cdot 5y^4}{2 \cdot 4 \cdot 6 \cdot 8x^7}, \text{ etc.}$$

**11.** Expand  $\sqrt{2}$ , or  $(1+1)^{\frac{1}{2}}$ , keeping the factors of the coefficients distinct.

$$\text{Ans. } 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{3}{2 \cdot 4 \cdot 6} - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}, \text{ etc.}$$

**422.** An *Infinite Series* must not be confounded with an *Infinite Quantity*.

**423.** An *Infinite Quantity* is a quantity so great that nothing can be *added* to it.

**424.** An *Infinite Series* is a series in which the *number of terms* is *unlimited*.

**425.** The *magnitude* of the former admits of *no increase*; while in the latter the *number of terms* admits of *no increase*, and yet the sum of all the terms may be a small quantity.

Thus, if the series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$ , etc., in which each succeeding term is *half* the preceding, is continued to *infinity*, the *sum* of all the terms cannot exceed a *unit*.

**426.** When one quantity continually approximates another without reaching it, the latter is called the *Limit* of the former.

**418. To Expand a Compound Surd into an Infinite Series**6. Reduce  $\sqrt{1+x}$  to an infinite series.

OPERATION.

$$\begin{array}{r}
 1 + x \left| 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}, \text{ etc.}, \text{ Ans.} \right. \\
 \hline
 2 + \frac{x}{2} \left| \begin{array}{l} 1 \\ + x \end{array} \right. \\
 \hline
 \phantom{2 + x - \frac{x^2}{8}} + x + \frac{x^2}{4} \\
 \hline
 2 + x - \frac{x^2}{8} \left| \begin{array}{l} - \frac{x^2}{4} \\ - \frac{x^3}{4} - \frac{x^3}{8} + \frac{x^4}{64} \end{array} \right. \\
 \hline
 2 + x - \frac{x^2}{4} + \frac{x^3}{16} \left| \begin{array}{l} + \frac{x^3}{8} - \frac{x^4}{64}, \text{ etc.} \end{array} \right. \text{ Hence, the}
 \end{array}$$

**RULE.**—*Extract the square root of the given surd (Art. 298.)*

7. Expand  $\sqrt{x^2 - y^2}$ . *Ans.*  $x - \frac{y^2}{2x} - \frac{y^4}{8x^3} - \frac{y^6}{16x^5}, \text{ etc.}$ 8. Expand  $\sqrt{2}$ , or  $\sqrt{1+1}$ . *Ans.*  $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16}, \text{ etc.}$ **419. The Binomial Theorem applied to the Formation of Infinite Series.**

The Binomial Theorem may often be employed with advantage, in finding the roots of binomials. For a root is expressed like a power, except the exponent of one is an *integer*, and that of the other is a *fraction*.

9. Expand  $(x + y)^{\frac{1}{2}}$  into an infinite series.

SOLUTION.—The terms without coefficients are

$$x^{\frac{1}{2}}, x^{-\frac{1}{2}}y, x^{-\frac{3}{2}}y^2, x^{-\frac{5}{2}}y^3, x^{-\frac{7}{2}}y^4, \text{ etc.}$$

The coefficient of the second term is  $+\frac{1}{2}$ ; of the 3d,  $\frac{\frac{1}{2} \times -\frac{1}{2}}{2} = -\frac{1}{8}$ ;of the 4th term,  $\frac{-\frac{1}{2} \times -\frac{3}{2}}{3} = +\frac{1}{16}, \text{ etc.}$ The series is  $x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y - \frac{1}{8}x^{-\frac{3}{2}}y^2 + \frac{1}{16}x^{-\frac{5}{2}}y^3, \text{ etc.}$ 

418. How expand a surd into an infinite series?

**420.** When the *index* of the required power of a binomial is a *positive integer*, the series will *terminate*. For, the *index* of the leading quantity continually decreases by 1; and soon becomes 0; then the series must stop. (Art. 269.)

**421.** When the index of the required power is *negative*, the series will never terminate. For, by the successive subtractions of a *unit* from the index, it will never become 0; and the series may be continued indefinitely.

10. Expand  $(x^2 + y)^{\frac{1}{2}}$  into an infinite series, keeping the factors of the coefficients distinct.

$$(x^2 + y)^{\frac{1}{2}} = x + \frac{y}{2x} - \frac{y^2}{2 \cdot 4x^3} + \frac{3y^3}{2 \cdot 4 \cdot 6x^5} - \frac{3 \cdot 5y^4}{2 \cdot 4 \cdot 6 \cdot 8x^7}, \text{ etc.}$$

11. Expand  $\sqrt{2}$ , or  $(1+1)^{\frac{1}{2}}$ , keeping the factors of the coefficients distinct.

$$\text{Ans. } 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{3}{2 \cdot 4 \cdot 6} - \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}, \text{ etc.}$$

**422.** An *Infinite Series* must not be confounded with an *Infinite Quantity*.

**423.** An *Infinite Quantity* is a quantity so great that nothing can be *added* to it.

**424.** An *Infinite Series* is a series in which the *number of terms* is *unlimited*.

**425.** The *magnitude* of the former admits of *no increase*; while in the latter the *number of terms* admits of *no increase*, and yet the sum of all the terms may be a small quantity.

Thus, if the series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$ , etc., in which each succeeding term is *half* the preceding, is continued to *infinity*, the *sum* of all the terms cannot exceed a *unit*.

**426.** When one quantity continually approximates another without reaching it, the latter is called the *Limit* of the former.

## CHAPTER XIX.

### LOGARITHMS.\*

**436.** The *Logarithm* of a number is the *exponent* of the power to which a given *fixed* number must be raised to produce that number.

**437.** This *Fixed Number* is called the *Base* of the system.

Thus, if 3 is the base, then 2 is the logarithm of 9, because  $3^2 = 9$ ; and 3 is the logarithm of 27, because  $3^3 = 27$ , and so on.

Again, if 4 is the base, then 2 is the logarithm of 16, because  $4^2 = 16$ ; and 3 is the logarithm of 64, because  $4^3 = 64$ , and so on.

**438.** In forming a *system* of logarithms, any number, except 1, may be taken as the base, and when the base is selected, all other numbers are considered as some *power* or *root* of this base. Hence, there may be an unlimited number of systems.

NOTE.—Since all *powers* and *roots* of 1 are 1, it is obvious that other numbers cannot be represented by its powers or roots. (Art. 289.)

**439.** There are two systems of logarithms in use, the Napierian system,† the base of which is 2.718281828, and the Common System, whose base is 10.‡

 The abbreviation *log.* stands for the term logarithm.

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436. What are logarithms? 437. What is this fixed number called? 439. Name the systems in use. The base of each.

\* The term *logarithm* is derived from two Greek words, meaning the *relation of numbers*.

† So called from Baron Napier, of Scotland, who invented logarithms in 1614.

‡ The common system was invented by Henry Briggs, an English mathematician, in 1624.

**440.** The *Base* of common logarithms being 10, all other numbers are considered as powers or roots of 10.

Thus, the log. of 1 is 0; for  $10^0$  equals 1 (Art. 259);  
 " " 10 is 1; for  $10^1$  " 10;  
 " " 100 is 2; for  $10^2$  " 100;  
 " " 1000 is 3; for  $10^3$  " 1000, etc. Hence,

The logarithm of any number between 1 and 10 is a fraction; for any number between 10 and 100, the logarithm is 1 plus a fraction; and for any number between 100 and 1000, the logarithm is 2 plus a fraction, and so on.

**441.** By means of negative exponents, this principle may be applied to fractions.

Thus (Art. 256), the log. of .1 is  $-1$ ; for  $10^{-1}$  equals .1;  
 " " .01 is  $-2$ ; for  $10^{-2}$  " .01;  
 " " .001 is  $-3$ ; for  $10^{-3}$  " .001.

Therefore, the logarithms for all numbers between 1 and 0.1 lie between 0 and  $-1$ , and are respectively equal to  $-1$  plus a fraction; for any number between 0.1 and 0.01, the logarithm is  $-2$  plus a fraction; and for any number between 0.01 and 0.001, the logarithm is  $-3$  plus a fraction, and so on.

Hence, the logarithms of all numbers greater than 10 or less than 1, and not exact powers of 10, are composed of two parts, an *integer* and a *fraction*.

Thus, the logarithm of 28 is 1.44716;  
 and of .28 is  $\bar{1}.44716$ .

**442.** The *integral* part of a logarithm is called the *Characteristic*; the *decimal* part, the *Mantissa*.

**443.** The *Characteristic* of the logarithm of a whole number is one less than the number of integral figures in the given number.

Thus, the characteristic of the logarithm of 49 is 1; that of 495 is 2; that of 4956 is 3; that of 6256.414 is also 3, etc.

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440. What is the logarithm of any number from 1 to 10? From 10 to 100? From 100 to 1000? 442. What is the integral part of a logarithm called? The decimal part? 443. What is the characteristic of the logarithm of a whole number?

**444.** The *Characteristic* of the logarithm of a decimal is *negative*, and is one *greater* than the number of ciphers before the first significant figure of the fraction.

Thus, the characteristic of the logarithm of  $\frac{1}{10}$  or .1 is  $-1$ ; that of  $\frac{1}{100}$ , or .01, is  $-2$ ; that of  $\frac{1}{1000}$ , or .001, is  $-3$ , etc. (Art. 256.)

The logarithm of .2 is  $-1$  with a decimal added to it; that of .05 is  $-2$  with a decimal added to it, etc.

NOTE.—It should be observed that the *characteristic only is negative*, while the mantissa, or *decimal part*, is always *positive*. To indicate this, the sign  $-$  is placed *over* the characteristic, instead of *before* it.

Thus, the logarithm of .2 is  $\bar{1}.30103$ ,  
 “ “ “ .05 is  $\bar{2}.69897$ , etc.

**445.** The *Decimal Part* of the logarithm of any number is the same as the logarithm of the number multiplied or divided by 10, 100, 1000, etc.

Thus, the logarithm of 1876 is 3.27325; of 18760 is 4.27325, etc.

## TABLES OF LOGARITHMS.

**446.** A *Table of Logarithms* is one which contains the logarithms of all numbers between given limits.

**447.** The Table found on the following pages gives the mantissas of common logarithms to five decimal places for all numbers from 1 to 1000, inclusive.

The *characteristics* are omitted, and must be supplied by inspection. (Arts. 443, 444.)

NOTES.—1. The first decimal figure in column 0 is often the same for several successive numbers, but is printed only once, and is understood to belong to each of the blank places below it.

2. The character (+) shows that the figure belonging to the place it occupies has changed from 9 to 0, and through the rest of this line the first figure of the mantissa stands in the next line below.

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444. What is the characteristic of the logarithm of a decimal? 445. What is the effect upon the decimal part of the log. of a number, if the number is multiplied or divided by 10, 100, 1000, etc. 446. What is a table of logarithms?

**448. To Find the *Logarithm* of any Number from 1 to 10.**

**RULE.**—*Look for the given number in the first line of the table ; its logarithm will be found directly below it.*

- |                             |                      |
|-----------------------------|----------------------|
| 1. Find the logarithm of 7. | <i>Ans.</i> 0.84510. |
| 2. Find the logarithm of 9. | <i>Ans.</i> 0.95424. |

**449. To Find the *Logarithm* of any Number from 10 to 1000, inclusive.**

**RULE.**—*Look in the column marked N for the first two figures of the given number, and for the third at the head of one of the other columns.*

*Under this third figure, and opposite the first two, will be found the last decimal figures of the logarithm. The first one is found in the column marked 0.*

*To this decimal prefix the proper characteristic. (Art. 443.)*

**NOTE.**—If the number contains 4 or more figures, multiply the tabular difference by the remaining figures, and rejecting from the right of the product as many figures as you multiply by, add the rest to the log. of the first 3 figures.

- |                                |                      |
|--------------------------------|----------------------|
| 3. Find the logarithm of 108.  | <i>Ans.</i> 2.03342. |
| 4. Find the logarithm of 176.  | <i>Ans.</i> 2.24551. |
| 5. Find the logarithm of 1999. | <i>Ans.</i> 3.30085. |

**450. To Find the *Logarithm* of a *Decimal Fraction*.**

**RULE.**—*Take out the logarithm of a whole number consisting of the same figures, and prefix to it the proper negative characteristic. (Art. 444.)*

**NOTE.**—If the number consists of an *integer* and a *decimal*, find the logarithm in the same manner as if *all* the figures were integers, and prefix the characteristic which belongs to the *integral* part. (Art. 443.)

- |                                 |                               |
|---------------------------------|-------------------------------|
| 6. What is the log. of 0.95 ?   | <i>Ans.</i> $\bar{1}.97772$ . |
| 7. What is the log. of 0.0125 ? | <i>Ans.</i> $\bar{2}.09691$ . |
| 8. What is the log. of 0.0075 ? | <i>Ans.</i> $\bar{3}.87506$ . |
| 9. What is the log. of 16.45 ?  | <i>Ans.</i> 1.21616.          |
| 10. What is the log. of 185.3 ? | <i>Ans.</i> 2.26787.          |

**451. To Find the Number belonging to a given Logarithm.**

**RULE.**—Look for the decimal figures of the given logarithm in the table under the column marked 0; and if all of them are not found in that column, look in the other columns on the right till you find them exactly, or very nearly; directly opposite, in the column marked *N*, will be found the first two figures, and at the top, over the logarithm, the third figure of the given number.

Make this number correspond to the characteristic of the given logarithm, by pointing off decimals, or by adding ciphers, if necessary, and it will be the number required.

**NOTE.**—If the characteristic of a logarithm is *negative*, the number belonging to it is a *fraction*, and as many ciphers must be prefixed to the number found in the table, as there are *units* in the characteristic *less 1*. (Art. 444.)

**452. When the Decimal Part of the given Logarithm is not exactly, or very nearly, found in the Table.**

**RULE.**—From the given logarithm subtract the next less logarithm found in the tables; annex ciphers to the remainder, and divide it by the tabular difference (marked *D*) as far as necessary.

To the number belonging to the less logarithm annex the quotient, and make the number thus produced correspond to the characteristic of the given logarithm, as above.

**NOTE.**—For every cipher annexed to the remainder, either a *significant figure* or a *cipher* must be put in the quotient.

- |  |                       |
|--|-----------------------|
| 11. What number belongs to 2.17231?          | <i>Ans.</i> 148.7.    |
| 12. What number belongs to 1.25261?          | <i>Ans.</i> 17.89.    |
| 13. What number belongs to 3.27715?          | <i>Ans.</i> 1893.     |
| 14. What number belongs to 2.30963?          | <i>Ans.</i> 204.      |
| 15. What number belongs to 4.29797?          | <i>Ans.</i> 19858.29. |
| 16. What number belongs to $\bar{1}.14488$ ? | <i>Ans.</i> 0.1396.   |
| 17. What number belongs to $\bar{2}.29136$ ? | <i>Ans.</i> 0.01956.  |
| 18. What number belongs to $\bar{3}.30928$ ? | <i>Ans.</i> 0.002038. |

**453.** Computations by logarithms are based upon the following principles:

1°. *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Let  $a$  and  $c$  denote any two numbers,  $m$  and  $n$  their logarithms, and  $b$  the base.

Then  $b^m = a$

And  $b^n = c$

Multiplying,  $b^{m+n} = ac.$

2°. *The logarithm of the dividend diminished by the logarithm of the divisor is equal to the logarithm of the quotient of the two numbers.*

Let  $a$  and  $c$  denote any two numbers,  $m$  and  $n$  their logarithms, and  $b$  the base.

Then  $b^m = a$

And  $b^n = c$

Dividing,  $b^{m-n} = a \div c.$

#### **454.** To *Multiply* by Logarithms.

1. Required the product of 35 by 23.

SOLUTION.—The log. of 35 = 1.54407

“ “ “ 23 = 1.36173

Adding, 2.90580. (Art. 453, Prin. 1.)

The number belonging is 805, *Ans.* Hence, the

**RULE**—*Add the logarithms of the factors; the sum will be the logarithm of the product.*

NOTES.—1. If the sum of the decimal parts exceeds 9, add the *tens* figure to the characteristic.

2. If either or all the characteristics are *negative*, they must be added according to Art. 65. But as the mantissa is always *positive*, that which is carried from the mantissa to the characteristic must be considered *positive*.

2. What is the product of 109.3 by 14.17 ?

3. What is the product of 1.465 by 1.347 ?

4. What is the product of .074 by 1500 ?

---

453. Upon what two principles are computations by logarithms based ? 454. How multiply by logarithms ?

**455. To Divide by Logarithms.**

5. Required the quotient of 120 by 15.

SOLUTION.—The log. of 120 = 2.07918

“ “ “ 15 = 1.17609

“ “ “ quotient = 0.90309. *Ans.* 8. Hence, the

**RULE.**—*From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.* (Art. 453, Prin. 2.)

**NOTES.**—1. When either of the characteristics is *negative*, or when the lower one is greater than the one above it, change the sign of the subtrahend, and proceed as in addition.

2. When 1 is carried from the mantissa to the characteristic, it must be considered *positive*, and be added to the characteristic *before* the sign is changed.

6. What is the quotient of 12.48 by 0.16?

7. What is the quotient of .045 by 1.20?

8. What is the quotient of 1.381 by .096?

**456.** *Negative quantities are divided in the same manner as positive quantities.*

If the sign of the divisor is the same as that of the dividend, prefix the sign + to the quotient; but if different, prefix the sign —.

9. Divide —128 by —47.

10. Divide —186 by —0.064.

11. Divide —0.156 by —0.86.

12. Divide —0.194 by 0.042.

**457. To Involve a Number by Logarithms.**

*Multiplication by logarithms is performed by addition.* (Art. 453.) Therefore, if the logarithm of a quantity is added to *itself once*, the result will be the logarithm of the *second power* of that quantity; if added to itself *twice*, the result will be the *third power* of that quantity, and so on. Hence, the

**RULE.**—*Multiply the logarithm of the number by the exponent of the required power.*

---

455. How divide by them? 457 How involve a number by logarithms?

NOTES.—1. This rule depends upon the principle that logarithms are the *exponents* of *powers* and *roots*, and a power or root is involved by multiplying its *index* into the *index* of the power required.

2. In this rule, whatever is carried from the mantissa to the characteristic is *positive*, whether the index itself is *positive* or *negative*.

13. What is the cube of 1.246.

SOLUTION.—The log. of 1.246 is 0.09551  
 Index of the required power is 3  
 Log. of power is 0.28653. *Ans.* 1.93435.

14. What is the fourth power of .135?

15. What is the tenth power of 1.42?

16. What is the twenty-fifth power of 1.234?

#### 458. To Extract the Root of a Number by Logarithms.

A quantity is resolved into any number of *equal factors*, by dividing its index into as many *equal parts*. (Art. 281.) Hence, the

RULE.—*Divide the logarithm of the number by the index of the required root.*

NOTE.—This rule depends upon the principle that the *root* of a quantity is found by *dividing the exponent* by the number expressing the required root. (Art. 296.)

17. What is the square root of 1.69?

SOLUTION.—The log. of 1.69 is 0.22789  
 The index is 2, 2 ) .22789  
 Logarithm of root, 0.11394 *Ans.* 1.3.

18. What is the cube root of 143.2?

19. What is the sixth root of 1.62?

20. What is the eighth root of 1549?

21. What is the tenth root of 1876?

459. If the characteristic of the logarithm is *negative*, and cannot be *divided* by the *index* of the required root without a *remainder*, make it *positive* by adding to the *characteristic* such a *negative* number as will make it *exactly divisible* by the *divisor*, and *prefix* an equal *positive* number to the decimal part of the logarithm.

22. It is required to find the cube root of .0164.

SOLUTION.—The log. of .0164 is  $\bar{2}.21484$ .

Preparing the log., 
$$\begin{array}{r} 3 \overline{) 3 + 1.21484} \\ \underline{1.40494.} \end{array}$$
 Ans. 0.25406+.

23. What is the sixth root of .001624 ?

24. What is the seventh root of .01449 ?

25. What is the eighth root of .0001236 ?

#### 460. To Calculate Compound Interest by Logarithms.

RULE.—Find the amount of 1 dollar for 1 year ; multiply its logarithm by the number of years, and to the product add the logarithm of the principal. The sum will be the logarithm of the amount for the given time.

From the amount subtract the principal, and the remainder will be the interest.

NOTES.—1. If the interest becomes due *half yearly* or *quarterly*, find the amount of one dollar for the half year or quarter, and multiply the logarithm by the number of half years or quarters in the time.

2. This rule is based upon the principle that the several *amounts* in compound interest form a *geometrical series*, of which the principal is the first term, the amount of \$1 for 1 year the ratio, and the number of years + 1 the number of terms

26. What is the amount of \$1565 for 40 years, at 6 per cent compound interest ?

SOLUTION.—The amt. of \$1 for 1 year is \$1.06 ; its log., 0.02531  
 The number of years,  $\underline{40}$   
 Product, 1.01240  
 The given principal, \$1565 ; its log.,  $\underline{3.19453}$   
 Ans. \$16103.78. 4.20693

27. What is the amount of \$1500, at 7 per cent compound interest, for 4 years ? Ans. \$1966.05.

28. What is the amount of \$370, at 5 per cent compound interest for 33 years ? Ans. \$1851.27+.

TABLE OF COMMON LOGARITHMS.

N.	0	1	2	3	4	5	6	7	8	9	D.
		.00000	.30103	.47712	.60206	.69897	.77815	.84510	.90309	.95424	
10	.00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	416
11	4139	4532	4922	5308	5691	6070	6446	6819	7188	7555	379
12	7918	8279	8636	8991	9342	9691	+037	+380	+721	1059	349
13	.11394	1727	2057	2385	2711	3033	3354	3672	3988	4302	322
14	4613	4922	5229	5534	5836	6137	6435	6732	7026	7319	301
15	7609	7898	8184	8469	8752	9033	9313	9590	9866	+140	281
16	.20412	0683	0952	1219	1484	1748	2011	2272	2531	2789	264
17	3045	3300	3553	3805	4055	4304	4551	4797	5042	5285	249
18	5527	5768	6007	6245	6482	6717	6951	7184	7416	7646	235
19	7875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	.30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	2222	2428	2634	2838	3041	3244	3445	3646	3846	4044	203
22	4242	4439	4635	4831	5025	5218	5411	5603	5794	5984	193
23	6173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	8021	8202	8382	8561	8739	8917	9094	9270	9445	9620	178
25	9794	9967	+140	+312	+483	+654	+824	+993	1162	1330	171
26	.41497	1664	1830	1996	2160	2325	2488	2651	2814	2975	165
27	3136	3297	3457	3616	3775	3933	4091	4248	4405	4560	158
28	4716	4871	5025	5179	5332	5485	5637	5788	5939	6090	153
29	6240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	.47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	143
31	9136	9276	9416	9554	9693	9831	9969	+106	+243	+379	138
32	.56515	0651	0786	0920	1055	1188	1322	1455	1587	1720	133
33	1851	1983	2114	2244	2375	2505	2634	2763	2892	3020	130
34	3148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	4407	4531	4654	4778	4900	5023	5145	5267	5388	5509	123
36	5630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	6820	6937	7054	7171	7287	7403	7519	7634	7749	7864	116
38	7978	8093	8206	8320	8433	8546	8659	8771	8883	8995	113
39	9107	9218	9329	9439	9550	9660	9770	9879	9988	+097	110
40	.60206	0314	0423	0531	0638	0746	0853	0959	1066	1172	108
41	1278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	2325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	3347	3448	3548	3649	3749	3849	3949	4048	4147	4247	100
44	4345	4444	4542	4640	4738	4836	4934	5031	5128	5225	98
45	5321	5418	5514	5610	5706	5801	5897	5992	6087	6181	95
46	6276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	7210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	8124	8215	8305	8395	8485	8574	8664	8753	8842	8931	89
49	9020	9108	9197	9285	9373	9461	9548	9636	9723	9810	88
N.	0	1	2	3	4	5	6	7	8	9	D.

## CHAPTER XIX.

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**436.** The *Logarithm* of a number is the *exponent* of the power to which a given *fixed* number must be raised to produce that number.

**437.** This *Fixed Number* is called the *Base* of the system.

Thus, if 3 is the base, then 2 is the logarithm of 9, because  $3^2 = 9$ ; and 3 is the logarithm of 27, because  $3^3 = 27$ , and so on.

Again, if 4 is the base, then 2 is the logarithm of 16, because  $4^2 = 16$ ; and 3 is the logarithm of 64, because  $4^3 = 64$ , and so on.

**438.** In forming a *system* of logarithms, any number, except 1, may be taken as the base, and when the base is selected, all other numbers are considered as some *power* or *root* of this base. Hence, there may be an unlimited number of systems.

NOTE.—Since all *powers* and *roots* of 1 are 1, it is obvious that other numbers cannot be represented by its powers or roots. (Art. 289.)

**439.** There are two systems of logarithms in use, the Napierian system,† the base of which is 2.718281828, and the Common System, whose base is 10.‡

 The abbreviation *log.* stands for the term logarithm.

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436. What are logarithms? 437. What is this fixed number called? 439. Name the systems in use. The base of each.

\* The term *logarithm* is derived from two Greek words, meaning the *relation* of *numbers*.

† So called from Baron Napier, of Scotland, who invented logarithms in 1614.

‡ The common system was invented by Henry Briggs, an English mathematician, in 1624.

**440.** The *Base* of common logarithms being 10, all other numbers are considered as powers or roots of 10.

Thus, the log. of 1 is 0; for  $10^0$  equals 1 (Art. 259);  
 " " 10 is 1; for  $10^1$  " 10;  
 " " 100 is 2; for  $10^2$  " 100;  
 " " 1000 is 3; for  $10^3$  " 1000, etc. Hence,

The logarithm of any number between 1 and 10 is a fraction; for any number between 10 and 100, the logarithm is 1 plus a fraction; and for any number between 100 and 1000, the logarithm is 2 plus a fraction, and so on.

**441.** By means of negative exponents, this principle may be applied to fractions.

Thus (Art. 256), the log. of .1 is  $-1$ ; for  $10^{-1}$  equals .1;  
 " " .01 is  $-2$ ; for  $10^{-2}$  " .01;  
 " " .001 is  $-3$ ; for  $10^{-3}$  " .001.

Therefore, the logarithms for all numbers between 1 and 0.1 lie between 0 and  $-1$ , and are respectively equal to  $-1$  plus a fraction; for any number between 0.1 and 0.01, the logarithm is  $-2$  plus a fraction; and for any number between 0.01 and 0.001, the logarithm is  $-3$  plus a fraction, and so on.

Hence, the logarithms of all numbers greater than 10 or less than 1, and not exact powers of 10, are composed of two parts, an *integer* and a *fraction*.

Thus, the logarithm of 28 is 1.44716;  
 and of .28 is  $\bar{1}.44716$ .

**442.** The *integral* part of a logarithm is called the *Characteristic*; the *decimal* part, the *Mantissa*.

**443.** The *Characteristic* of the logarithm of a whole number is one less than the number of integral figures in the given number.

Thus, the characteristic of the logarithm of 49 is 1; that of 495 is 2; that of 4956 is 3; that of 6256.414 is also 3, etc.

440. What is the logarithm of any number from 1 to 10? From 10 to 100? From 100 to 1000? 442. What is the integral part of a logarithm called? The decimal part? 443. What is the characteristic of the logarithm of a whole number?

He takes another example of two quantities, and finds the statement true; then another, and so on. Hence, he concludes the proposition is a universal truth.

**465.** Suppose this proposition were enunciated:

*The sum of any number of terms of the arithmetical series 1, 3, 5, 7, etc., to  $n$  terms, is equal to  $n^2$ .*

We see by inspection that the sum of the first two terms,  $1 + 3 = 4$ , or  $2^2$ ; that the sum of the first three terms,  $1 + 3 + 5 = 9$ , or  $3^2$ ; that the sum of four terms,  $1 + 3 + 5 + 7 = 16$ , or  $4^2$ , and so on. Hence, we may conclude that the *proposition is true* if the series be extended indefinitely. Or,

Since we know the *proposition is true* when  $n$  denotes a small number of terms, and that the value of any term in the series, as the 5th, 7th, 9th, etc., is equal to  $2n - 1$ , we may suppose for this value of  $n$ , that

$$1 + 3 + 5 + 7 \dots + (2n - 1) = n^2. \quad (1)$$

Adding  $2n + 1$  to both members, we have

$$1 + 3 + 5 + 7 \dots + (2n - 1) + (2n + 1) = n^2 + 2n + 1. \quad (2)$$

Factoring second member of (2),  $n^2 + 2n + 1 = (n + 1)^2$ .

Therefore, since the sum of  $n$  terms of the series  $= n^2$ , it follows that the sum of  $n + 1$  terms  $= (n + 1)^2$ , and so on. Hence, the proposition must be *universally true*.

**466.** In Geometry, we have the proposition:

*The sum of the three angles of a triangle is equal to two right angles.*

We find it to be true in one case; then in another, etc. Hence, we conclude the *proposition is universally true*.

NOTES.—1. It is sometimes objected that *this method of proof* is less satisfactory to the learner than a *more rigorous* process of reasoning. But when it is fully understood, it is believed that it will produce the fullest conviction of the truths designed to be established.

2. In metaphysics and the natural sciences, the term *induction* is applied to the *assumption* that certain laws are *general* which by experiment have been proved to be true in certain cases. But we cannot be *sure* that these laws hold for any cases except those which have been examined, and can never arrive at the conclusion that they are *necessary* truths.

**448. To Find the *Logarithm* of any Number from 1 to 10.**

**RULE.**—Look for the given number in the first line of the table; its logarithm will be found directly below it.

- |                             |                      |
|-----------------------------|----------------------|
| 1. Find the logarithm of 7. | <i>Ans.</i> 0.84510. |
| 2. Find the logarithm of 9. | <i>Ans.</i> 0.95424. |

**449. To Find the *Logarithm* of any Number from 10 to 1000, inclusive.**

**RULE.**—Look in the column marked *N* for the first two figures of the given number, and for the third at the head of one of the other columns.

Under this third figure, and opposite the first two, will be found the last decimal figures of the logarithm. The first one is found in the column marked 0.

To this decimal prefix the proper characteristic. (Art. 443.)

**NOTE.**—If the number contains 4 or more figures, multiply the tabular difference by the remaining figures, and rejecting from the right of the product as many figures as you multiply by, add the rest to the log. of the first 3 figures.

- |                                |                      |
|--------------------------------|----------------------|
| 3. Find the logarithm of 108.  | <i>Ans.</i> 2.03342. |
| 4. Find the logarithm of 176.  | <i>Ans.</i> 2.24551. |
| 5. Find the logarithm of 1999. | <i>Ans.</i> 3.30085. |

**450. To Find the *Logarithm* of a *Decimal Fraction*.**

**RULE.**—Take out the logarithm of a whole number consisting of the same figures, and prefix to it the proper negative characteristic. (Art. 444.)

**NOTE.**—If the number consists of an *integer* and a *decimal*, find the logarithm in the same manner as if *all* the figures were integers, and prefix the characteristic which belongs to the *integral* part. (Art. 443.)

- |                                |                               |
|--------------------------------|-------------------------------|
| 6. What is the log. of 0.95?   | <i>Ans.</i> $\bar{1}.97772$ . |
| 7. What is the log. of 0.0125? | <i>Ans.</i> $\bar{2}.09691$ . |
| 8. What is the log. of 0.0075? | <i>Ans.</i> $\bar{3}.87506$ . |
| 9. What is the log. of 16.45?  | <i>Ans.</i> 1.21616.          |
| 10. What is the log. of 185.3? | <i>Ans.</i> 2.26787.          |

**451. To Find the Number belonging to a given Logarithm.**

**RULE.**—Look for the decimal figures of the given logarithm in the table under the column marked 0; and if all of them are not found in that column, look in the other columns on the right till you find them exactly, or very nearly; directly opposite, in the column marked *N*, will be found the first two figures, and at the top, over the logarithm, the third figure of the given number.

Make this number correspond to the characteristic of the given logarithm, by pointing off decimals, or by adding ciphers, if necessary, and it will be the number required.

**NOTE.**—If the characteristic of a logarithm is *negative*, the number belonging to it is a *fraction*, and as many ciphers must be prefixed to the number found in the table, as there are *units* in the characteristic *less 1*. (Art. 444.)

**452. When the Decimal Part of the given Logarithm is not exactly, or very nearly, found in the Table.**

**RULE.**—From the given logarithm subtract the next less logarithm found in the tables; annex ciphers to the remainder, and divide it by the tabular difference (marked *D*) as far as necessary.

To the number belonging to the less logarithm annex the quotient, and make the number thus produced correspond to the characteristic of the given logarithm, as above.

**NOTE.**—For every cipher annexed to the remainder, either a *significant figure* or a *cipher* must be put in the quotient.

- |                                     |                       |
|-------------------------------------|-----------------------|
| 11. What number belongs to 2.17231? | <i>Ans.</i> 148.7.    |
| 12. What number belongs to 1.25261? | <i>Ans.</i> 17.89.    |
| 13. What number belongs to 3.27715? | <i>Ans.</i> 1893.     |
| 14. What number belongs to 2.30963? | <i>Ans.</i> 204.      |
| 15. What number belongs to 4.29797? | <i>Ans.</i> 19858.29. |
| 16. What number belongs to 1.14488? | <i>Ans.</i> 0.1396.   |
| 17. What number belongs to 2.29136? | <i>Ans.</i> 0.01956.  |
| 18. What number belongs to 3.30928? | <i>Ans.</i> 0.002038. |

**453.** Computations by logarithms are based upon the following principles:

1°. *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Let  $a$  and  $c$  denote any two numbers,  $m$  and  $n$  their logarithms, and  $b$  the base.

$$\begin{array}{ll} \text{Then} & b^m = a \\ \text{And} & b^n = c \\ \text{Multiplying,} & b^{m+n} = ac. \end{array}$$

2°. *The logarithm of the dividend diminished by the logarithm of the divisor is equal to the logarithm of the quotient of the two numbers.*

Let  $a$  and  $c$  denote any two numbers,  $m$  and  $n$  their logarithms, and  $b$  the base.

$$\begin{array}{ll} \text{Then} & b^m = a \\ \text{And} & b^n = c \\ \text{Dividing,} & b^{m-n} = a \div c. \end{array}$$

#### **454. To Multiply by Logarithms.**

1. Required the product of 35 by 23.

$$\begin{array}{rcl} \text{SOLUTION.—The log. of 35} & = & 1.54407 \\ \text{“ “ “ 23} & = & 1.36173 \\ \text{Adding,} & & 2.90580. \quad (\text{Art. 453, Prin. 1.}) \\ \text{The number belonging is 805, Ans.} & & \text{Hence, the} \end{array}$$

**RULE**—*Add the logarithms of the factors; the sum will be the logarithm of the product.*

**NOTES.**—1. If the sum of the decimal parts exceeds 9, add the *tens* figure to the characteristic.

2. If either or all the characteristics are *negative*, they must be added according to Art. 65. But as the mantissa is always *positive*, that which is carried from the mantissa to the characteristic must be considered *positive*.

2. What is the product of 109.3 by 14.17 ?
3. What is the product of 1.465 by 1.347 ?
4. What is the product of .074 by 1500 ?

$$\text{FORMULA.} \quad t = \frac{1}{r}.$$

**RULE.**—Divide 1 by the rate, and the quotient will be the time required.

11. How long will it take \$1500, at 5 per cent, to double itself?

$$\text{SOLUTION.} \quad t = \frac{1}{r} = \frac{1}{.05} = 20. \quad \text{Ans. 20 years.}$$

12. How long will it take \$680, at 6 per cent, to double itself?

13. How long will it take \$8475, at 10 per cent, to double itself?

**477. To Find the Rate at which any Principal, at Simple Interest, will Double itself in a Given Time.**

$$\text{By the preceding formula,} \quad t = \frac{1}{r}.$$

Multiplying by  $\frac{r}{t}$ , we have the

$$\text{FORMULA.} \quad r = \frac{1}{t}.$$

**RULE.**—Divide 1 by the time; the quotient will be the rate per cent required.

[For other formulas in Simple Interest, see Arts. 242–246.]

14. If \$1700 doubles itself in 8 years, what is the rate?

Ans.  $12\frac{1}{2}$  per cent.

15. At what rate per cent will \$5000 double itself in 40 years?

## COMPOUND INTEREST.

**478. Interest** may be compounded *annually, semi-annually, quarterly*, etc. It is understood to be compounded annually, unless otherwise mentioned.

**479. To Find the Amount of a given Principal at Compound Interest, the Rate and Time being given.**

Let  $p$  denote the principal,  $r$  the rate,  $n$  the number of years, and  $a$  the amount.

Since the amount equals the principal plus the interest, it follows that the amount of \$1 for 1 year equals  $1+r$ ; therefore,  $p(1+r)$  equals the amount of  $p$  dollars for 1 year, which is the principal for the second year.

Again, the amount of this new principal  $p(1+r)$  for 1 year =  $p(1+r)(1+r) = p(1+r)^2$ , which is the amount of  $p$  dollars for two years.

In like manner,  $p(1+r)^3$  is the amount of  $p$  dollars for three years and so on, forming a geometrical series, of which the principal  $p$  is the first term,  $1+r$  the ratio, and the number of years  $+ 1$ , the number of terms. The terms of the series are

$$p, p(1+r), p(1+r)^2, p(1+r)^3, p(1+r)^4, \dots p(1+r)^n.$$

The last term,  $p(1+r)^n$ , is the amount of  $p$  dollars for  $n$  years. Hence, the

$$\text{FORMULA.} \quad a = p(1+r)^n.$$

**RULE.**—*Multiply the principal by the amount of \$1 for 1 year, raised to a power denoted by the number of years; the product will be the amount.*

#### 480. To Find the Compound Interest for the given Time and Rate.

*Subtract the principal from the amount, and the remainder will be the compound interest.*

**NOTE.**—When the number of years or periods is large, the operations are shortened by using logarithms.

16. What is the amount of \$842, at 6 per cent compound interest, for 4 years?

$$\text{SOLUTION.} \quad \$842 \times (1.06)^4 = \$1063, \text{ Ans.}$$

17. What is the amount of \$1500, at 5 per cent for 6 yrs., compound interest?

**481.** If the interest is compounded *semi-annually*,  $\frac{r}{2}$  will denote the interest of \$1 for a half year. Then, at compound interest, the amount of  $p$  dollars for  $n$  years is

$$p \left( 1 + \frac{r}{2} \right)^{2n}.$$

**482.** If the interest is compounded *quarterly*, then  $\frac{r}{4}$  will denote the interest of \$1 for a quarter. Then, at compound interest, the amount of  $p$  dollars for  $n$  years is

$$p \left(1 + \frac{r}{4}\right)^{4n}, \text{ etc.}$$

18. What is the amount of \$2000, for 3 years at 6 per cent, compounded semi-annually? *Ans.* \$2388.05.

19. What is the amount of \$5000 for 2 years, at 4 per ct., compounded quarterly?

**483.** By transposing, factoring, etc., the formula in Art. 479, we have,

The first term,

$$p = \frac{a}{(1+r)^n}.$$

The number of terms,

$$n = \frac{\log. a - \log. p}{\log. (1+r)}.$$

The ratio,

$$r = \left(\frac{a}{p}\right)^{\frac{1}{n}} - 1.$$

## DISCOUNT.

**484.** *Discount* is an allowance made for the payment of money before it is due.

**485.** The *Present Worth* of a debt payable at a future time is the *sum* which, if put at legal interest, will *amount* to the debt in the given time.

**486.** To Find the *Present Worth* of a Sum at Simple Interest, the Time of Payment and the Rate being given.

Let  $s$  denote the sum due,  $n$  the number of years, and  $r$  the interest of \$1 for 1 year.

Since  $r$  is the interest of \$1 for 1 year,  $nr$  must be the interest for  $n$  years, and  $1+nr$  the amount of \$1 for  $n$  years. Therefore,  $s+(1+nr)$  is the present worth of the given sum.

Putting  $P$  for present worth, we have the

$$\text{FORMULA.} \quad P = \frac{s}{1+nr}.$$

**RULE.**—Divide the sum due by the amount of \$1 for the given time and rate; the quotient will be the present worth.

**487.** To Find the *Discount*, the *Present Worth* being given.

*Subtract the present worth from the debt.*

20. What is the present worth of \$2500 payable in 4 yrs., interest being 7 per cent? What the discount?

$$\begin{aligned} \text{SOLUTION. } P &= \frac{s}{1+nr} = \frac{\$2500}{1.28} = \$1953.125, \text{ pres. worth;} \\ \$2500 - \$1953.125 &= \$546.875, \text{ discount,} \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{SOLUTION. } P &= \frac{s}{1+nr} = \frac{\$2500}{1.28} = \$1953.125, \text{ pres. worth;} \\ \$2500 - \$1953.125 &= \$546.875, \text{ discount,} \end{aligned}} \right\} \text{Ans.}$$

21. What is the present worth of \$3600 due in 5 years, at 6 per cent? What is the discount?

22. Find the present worth of \$7800 due in 6 years, interest 5 per cent? What the discount?

## COMPOUND DISCOUNT.

**488.** To Find the *Present Worth* of a Sum at Compound Interest, the Time and the Rate being given.

Let  $s$  denote the sum due,  $n$  the number of years,  $r$  the rate per ct.

Since  $r$  is the rate,  $1+r$  is the amount of \$1 for 1 year; then the amount for  $n$  years compound interest is  $(1+r)^n$ . (Art. 479.) That is, \$1 is the present worth of  $(1+r)^n$  due in  $n$  years. Therefore,  $s + (1+r)^n$  must be the present worth of the given sum.

Putting  $P$  for the present worth, we have the

$$\text{FORMULA. } P = \frac{s}{(1+r)^n}.$$

**RULE.**—*Divide the sum due by the amount of \$1 at compound interest, for the given time and rate; the quotient will be the present worth.*

23. What is the present worth of \$1000 due in 4 years, at 5 per cent compound interest?

$$\text{SOLUTION. } P = \frac{s}{(1+r)^n} = \frac{\$1000}{(1.05)^4} = \$822.71, \text{ Ans.}$$

24. What is the present worth of \$2300 due in 5 years, at 6 per cent compound interest?

## COMMERCIAL DISCOUNT.

**489. Commercial Discount** is a per cent taken from the face of bills, the marked price of goods, etc., without regard to time.

**490. To Find the Commercial Discount on a Bill of Goods, the Face of the Bill and the Per Cent Discount being given.**

Let  $b$  denote the base or face of the bill,  $r$  the rate, and  $d$  the discount or percentage.

Then  $b \times r$  will be the discount. Hence, the

$$\text{FORMULA.} \quad d = br.$$

**RULE.**—*Multiply the face of the bill by the given rate, and the product will be the commercial discount.*

**491. To Find the Cash Value or Net Proceeds of a Bill.**

*Subtract the commercial discount from the face of the bill.*

25. Required discount and net value of a bill of goods amounting to \$960, on 90 days, at  $12\frac{1}{2}$  per cent off for cash?

**SOLUTION.**  $\$960 \times .125 = \$120$ , discount;  $\$960 - \$120 = \$840$ , Ans.

26. Required the cash value of a bill amounting to \$2500, the discount being 10 per cent, and 5 per cent off for cash.

**492. To Mark Goods so as to allow a Discount, and make any proposed per cent Profit.**

Let  $c$  denote the cost,  $r$  the per cent profit, and  $d$  the per cent disc.

Since  $r$  is the per cent profit,  $1 + r$  is the selling price of \$1 cost, and  $c(1 + r)$  the selling price of  $c$  dollars cost.

Again, since  $d$  is the per cent discount from the marked price, and the marked price is 100 per cent of itself,  $1 - d$  must be the net value of \$1 marked price. Therefore,

$$c(1 + r) + (1 - d) = \text{the marked price.}$$

Putting  $m$  for the marked price, we have the

$$\text{FORMULA.} \quad m = \frac{c(1 + r)}{1 - d}.$$

**RULE.**—*Multiply the cost by 1 plus the per cent gain, and divide the product by 1 minus the proposed discount.*

27. A trader paid \$25 for a package of goods; at what price must it be marked that he may deduct 5 per cent, and yet make a profit of 10 per cent?

SOLUTION.  $m = \frac{c(1+r)}{1-d} = \frac{\$25 \times 1.10}{.95} = \$28.94+, \text{ Ans.}$

28. A merchant buys a case of silks at \$1.75 a yard; what must he mark them that he may deduct 10 per cent, and yet make 20 per cent?

29. A grocer bought flour at \$6½ a barrel; what price must he mark it that he may fall 3 per cent, and leave a profit of 25 per cent?

### INVESTMENTS.

493. The *Value of an Investment* in National and State securities, Railroad Bonds, etc., depends upon their market value, the rate of interest they bear, and the certainty of payment.

494. The *Dividends* of stocks and bonds are reckoned at a certain per cent on the par value of their shares, which is commonly \$100.

495. To Find the *Per Cent* which an Investment will pay, the Cost of a Share and the Rate of Dividend being given.

Let  $c$  denote the cost or market value of 1 share of stock,  $p$  its par value, and  $r$  the annual rate of dividend.

Since  $r$  is the rate of dividend and  $p$  the par value,  $pr$  must be the dividend on 1 share for 1 year. Therefore,  $pr + c$  will be the per cent received on the cost of 1 share. (Art. 238.)

Putting  $R$  for the per cent received on the cost of 1 share, we have the

FORMULA.  $R = \frac{pr}{c}.$

RULE.—Find the dividend on the given shares at the given rate, and divide this by the cost; the quotient will be the per cent received on the investment.

NOTE.—When the stock is *above* or *below* par, the *premium* or *discount* must be added to or subtracted from its par value to give the cost.

30. What per cent interest does a man receive on an investment of \$5000 in the Bank of Commerce, its dividends being 10 per cent, and the shares 5 per cent above par?

SOLUTION.—The premium on the stock =  $\$5000 \times .05 = \$250$ .  
Therefore, the cost =  $\$5000 + \$250 = \$5250$ .

Again, the dividend on stock =  $\$5000 \times .10 = \$500$ . Therefore,  $\$500 + \$5250 = 9\frac{1}{2}\%$  per cent, *Ans.*

31. A invested \$6000 in New York 6 per cent bonds, at 3 per cent premium. What per cent did he receive on his investment?

32. A man lays out \$1000 in Alabama 10 per cents, at a discount of 20 per cent. What per cent did he receive on his investment?

33. What per cent will a man receive on 50 shares of Pennsylvania Railroad stock, the premium being 4 per ct., and the dividend 10 per cent?

34. Which are preferable, Massachusetts 6 per cent bonds at par, or Ohio 8 per cent bonds at 2 per cent premium?

**496. To Find the *Amount* of a given Remittance which a Factor can Invest, and Reserve a Specified Per Cent for his Commission.**

Let  $s$  denote the sum remitted, and  $r$  the per cent commission.

The sum remitted includes both the sum invested and the commission. Now \$1 remitted is 100 per cent, or once itself; and adding the per cent to it, we have  $1+r$ , the cost of \$1 invested. Therefore,  $s \div (1+r)$  must be the amount invested.

Putting  $a$  for the amount invested, we have the

$$\text{FORMULA.} \quad a = \frac{s}{1+r}.$$

**RULE.**—*Divide the remittance by 1 plus the per cent commission; the quotient will be the amount to invest.*

35. A clergyman remitted to his agent \$2500 to purchase books. After deducting 4 per cent commission, how much does he lay out in books?

SOLUTION.  $a = \frac{s}{1+r} = \frac{\$2500}{1.04} = \$2403.85$ , Ans.

36. A gentleman remitted \$25000 to a broker, to be invested in stocks. After deducting  $1\frac{1}{2}$  per cent, how much did he invest, and what was his commission?

### SINKING FUNDS.

497. *Sinking Funds* are sums of money set apart or deposited annually, for the payment of public debts, and for other purposes.

#### CASE I

498. To Find the *Amount* of an Annual Deposit at Compound Interest, the Rate and Time being given.

Let  $s$  denote the annual deposit or sum set apart,  $r$  the rate per cent,  $n$  the number of years, and  $a$  the amount required.

Since the same sum is deposited at the end of each year, and put at compound interest, it follows that the deposit at the end of the

$$1\text{st year} = s$$

$$2\text{d} \quad " = s + s(1+r)$$

$$3\text{d} \quad " = s + s(1+r) + s(1+r)^2$$

$$n\text{th} \quad " = s + s(1+r) + s(1+r)^2 \dots + s(1+r)^{n-1},$$

forming a geometrical series; the annual deposit being the first term, the amount of \$1 for 1 year the ratio, the number of years the number of terms, and the annual deposit multiplied by the amount of \$1 for 1 year, raised to that power whose index is 1 less than the number of years, the last term; and the amount is equal to the sum of the series. (Art. 402.) Hence, we have the

$$\text{FORMULA.} \quad a = \frac{(1+r)^n - 1}{r} s.$$

RULE.—Multiply the amount of \$1 annual deposit for the given time and rate by the given annual deposit; the product will be the amount required.

37. A clerk annually deposited \$150 in a savings bank which pays 6 per cent compound interest. What amount will be due him in 5 years?

SOLUTION.  $a = \frac{(1+r)^n - 1}{r} s = \frac{(1.06)^5 - 1}{.06} \$150 = \$845.75, \text{ Ans.}$

38. A man agrees to give \$300 annually to build a church. What will his subscription amount to in 4 years, at 7 per cent compound interest?

39. If a teacher lays up \$500 annually, and puts it at 5 per cent compound interest for 10 years, how much will he be worth?

### CASE II.

499. To Find the *Annual Deposit* required to produce a given Amount at Compound Interest, the Rate and Time being given.

By the formula in the preceding article, we have

$$\frac{(1+r)^n - 1}{r} s = a.$$

Dividing by coefficient of  $s$ , we have the

FORMULA.  $s = a \div \frac{(1+r)^n - 1}{r}.$

RULE.—*Divide the amount to be raised by the amount of \$1 annual deposit for the given time and rate; the quotient will be the annual deposit required.*

NOTE.—To cancel the debt at maturity, the sum set apart as a sinking fund is supposed to be put at compound interest for the given time and rate.

40. A father promises to give his daughter \$5000 as a wedding present. Suppose the event to occur in 5 years, what sum must he annually deposit in a Trust Company, at 5 per cent compound interest, to meet his engagement?

SOLUTION.  $s = a + \frac{(1+r)^n - 1}{r} = \$5000 + \frac{(1.05)^5 - 1}{.05} = \frac{\$250}{.276} = \$905.80, \text{ Ans.}$

41. A man having lost his patrimony of \$20000, wishes to know how much must be annually deposited at 10 per cent, to recover it in 5 years?

42. A county borrows \$30000, at 6 per cent compound interest, to build a court-house; what sum must be set apart annually as a sinking fund to cancel the debt in 10 years?

## ANNUITIES.

**500.** *Annuities* are sums of money payable annually, or at regular intervals of time. They are computed according to the principles of compound interest.

### CASE I.

**501.** To Find the *Amount* of an Unpaid Annuity at Compound Interest, the Time and Rate per cent being given.

Let  $a$  be the annuity,  $1 + r$  the amount of \$1 for 1 year, and  $n$  the number of years.

The amount due at the end of the

1st year =  $a$ ,

2d " =  $a + a(1+r)$

3d " =  $a + a(1+r) + a(1+r)^2$ ,

4th " =  $a + a(1+r) + a(1+r)^2 + a(1+r)^3$ ,

$n$ th year =  $a + a(1+r) + a(1+r)^2 + a(1+r)^3 \dots a(1+r)^{n-1}$ .

forming a geometrical progression, the annuity being the first term, the amount of \$1 for 1 year the ratio, the number of years the number of terms, and the annuity multiplied by the amount of \$1 for 1 year raised to that power whose index is 1 less than the number of years, the last term. Therefore, the amount is equal to the sum of the series.

Putting  $S$  for the amount (Art. 498), we have the

$$\text{FORMULA.} \quad S = \frac{(1+r)^n - 1}{r} a.$$

**RULE.**—Multiply the amount of \$1 annuity, for the given time and rate, by the given annuity.

NOTE.—Logarithms may be used to advantage in some of the following examples.

43. What is due on an annuity of \$650, unpaid for 4 years, at 7 per cent compound interest?

$$\text{SOLUTION. } S = \frac{(1+r)^n - 1}{r} a = \frac{(1.07)^4 - 1}{.07} \$650 = \$2886, \text{ Ans.}$$

44. An annual pension of \$880 was unpaid for 6 years; what did it amount to at 6 per cent compound interest?

45. An annual tax of \$340 was unpaid for 7 years; what was due on it at 5 per cent compound interest?

## CASE II.

**502. To Find the *Present Worth* of an Annuity at Compound Interest, the Time of Continuance and the Rate being given.**

Let  $P$  denote the present worth; then the amount of  $P$  in  $n$  years will be equal to the amount of the annuity for the same time. Therefore,

$$P(1+r)^n = \frac{(1+r)^n - 1}{r} a.$$

Dividing each member by  $(1+r)^n$  (Art. 279), we have the

$$\text{FORMULA. } P = \frac{1 - (1+r)^{-n}}{r} a.$$

NOTE.—In applying the formula, the *negative* exponent may be made positive by *transferring* the quantity which it affects from the *numerator* to the *denominator* (Art. 279).

$$\text{Thus, } P = \frac{1 - (1+r)^{-n}}{r} a = \frac{1 - \frac{1}{(1+r)^n}}{r} a.$$

**RULE.**—Multiply the present worth of an annuity of \$1 for the given time by the given annuity.

46. What is the present worth of an annuity of \$375 for 6 years, at 7 per cent compound interest?

$$\text{SOLUTION. } P = \frac{1 - (1+r)^{-n}}{r} a = \frac{1 - (1.07)^{-6}}{.07} \$375 = \frac{1 - .66}{.07} \$375 = \$1785.71, \text{ Ans.}$$

47. What is the present worth of an annual pension of \$525 for 5 years, at 4 per cent compound interest?

CASE III.

**503. To Find the *Present Worth* of a Perpetual Annuity, the Rate being given.**

Let  $\infty$  denote infinity, then reducing the formula in Art. 502, we have this

$$\text{FORMULA. } P = \frac{a}{r}. \quad (\text{Art. 435.})$$

**RULE.**—*Divide the annuity by the interest of \$1 for 1 year, at the given rate.*

48. What is the present worth of a perpetual scholarship that pays \$150 annually, at 7 per cent compound interest?

$$\text{SOLUTION. } P = \frac{a}{r} = \frac{\$150}{.07} = \$2142.86, \text{ Ans.}$$

49. What is the present worth of a perpetual ground rent of \$850 a year, at 6 per cent?

CASE IV.

**504. To Find the *Present Worth* of an Annuity, commencing in a given Number of Years, the Rate and Time of Continuance being given.**

Let  $n$  be the number of years before it will commence, and  $N$  the number of years it is to continue. Then,

$$P = a \frac{1 - (1 + r)^{-(n+N)}}{r} - a \frac{1 - (1 + r)^{-n}}{r}.$$

Performing the subtraction indicated, we have the

$$\text{FORMULA. } P = \frac{a}{r} [(1 + r)^{-n} - (1 + r)^{-n-N}].$$

**RULE.**—*Find the present worth of the given annuity to the time it terminates; from this subtract its present worth to the time it commences.*

50. What is the present worth of an annuity of \$600, to commence in 4 years and to continue 12 years, at 7 per cent interest?

$$\text{SOLUTION. } P = \frac{a}{r} [(1 + r)^{-n} - (1 + r)^{-n-N}],$$

$$P = \frac{\$600}{.07} [(1.07)^{-4} - (1.07)^{-4-12}],$$

$$P = \frac{\$600 \times .4242}{.07} = \$3636, \text{ Ans.}$$

51. A father left an annual rent of \$2500 to his son for 6 years, and the reversion of it to his daughter for 12 years. What is the present worth of her legacy at 6 per cent interest?

### CASE V.

**505.** To Find the *Annuity*, the *Present Worth*, the *Time* and *Rate* being given.

By the formula in Article 502,

$$P = \frac{1 - (1 + r)^{-n}}{r} a.$$

Dividing by the coefficient of  $a$ , we have the

$$\text{FORMULA.} \quad a = \frac{Pr}{1 - (1 + r)^{-n}}.$$

**RULE.**—Divide the *present worth* by the *present worth of an annuity of \$1 for the given time and rate*.

52. The present worth of a pension, to continue 20 years at 6 per cent interest, is \$668. Required the pension.

$$\text{SOLUTION.} \quad a = \frac{Pr}{1 - (1 + r)^{-n}} = \frac{\$668 \times .06}{1 - (1.06)^{-20}} = \frac{\$668 \times .06}{.6882} = \$58.23, \text{ Ans.}$$

53. The present worth of an annuity, to continue 30 years at 5 per cent interest, is \$3840. What is the annuity?

**NOTE.**—The process of constructing formulas or rules, it will be seen, is based upon the principles of generalization combined with those of algebraic notation. The student will find it a profitable exercise to form others applicable to different classes of problems.

## CHAPTER XXI.

### DISCUSSION OF PROBLEMS.

**506.** The *Discussion* of a Problem consists in assigning all the different values possible to the arbitrary quantities which it contains, and interpreting the results.

**507.** An *Arbitrary Quantity* is one to which any value may be assigned at pleasure.

**PROBLEM.**—If  $b$  is subtracted from  $a$ , by what number must the remainder be multiplied that the product may be equal to  $c$ ?

Let  $x$  = the number.

Then  $(a - b)x = c$ .

Therefore,  $x = \frac{c}{a - b}$ .

**508.** The result thus obtained may have five different forms, depending on the relative values of  $a$ ,  $b$ , and  $c$ . To represent these forms, let  $m$  denote the multiplier.

I. Suppose  $a$  is *greater* than  $b$ . In this case  $a - b$  is *positive*, and  $c$  being *positive*, the quotient is *positive*. (Art. 112.) Consequently, the required multiplier must be *positive*, and the value of  $x$  will be of the form of  $+m$ .

II. Suppose  $a$  is *less* than  $b$ . In this case  $a - b$  is *negative*, and  $c$  divided by  $a - b$  is *negative*. (Art. 112.) Hence, the required multiplier must be *negative*, and the value of  $x$  is of the form of  $-m$ .

III. Suppose  $a$  is *equal* to  $b$ . In this case  $a - b = 0$ . Therefore, the value of  $x$  is of the form of  $\frac{m}{0}$ , or  $x = \frac{c}{0} = \infty$ . (Art. 434.)

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506. In what does the discussion of problems consist? 507. What is an arbitrary quantity?

IV. Suppose  $c$  is  $a$ , and  $a$  is either *greater* or *less* than  $b$ . In this case the value of  $x$  has the form  $\frac{0}{m}$ , or  $x = 0$ .

V. Suppose  $c$  equals  $a$ , and  $a$  equals  $b$ . In this case the value of  $x$  has the form  $\frac{0}{0}$ .

NOTE.—The student can easily test these principles by substituting numbers for  $a$ ,  $b$ , and  $c$ .

509. The Discussion of Problems may be further illustrated by the solution of the celebrated

### PROBLEM OF THE COURIERS.\*

Two couriers A and B, were traveling along the same road in the same direction, from C toward Q; A going at the rate of  $m$  miles an hour, and B  $n$  miles an hour. At 12 o'clock A was at a certain point  $P$ ; and B  $d$  miles in advance of A, in the direction of Q. When and where were they together?



This problem is general: we do not know from the statement whether the couriers were together before or after 12 o'clock, nor whether the place of meeting was on the right or the left of  $P$ .

Suppose the required time to be *after* 12 o'clock. Then the time after 12 is *positive*, and the time before 12 is *negative*; also, the distance reckoned from  $P$  toward Q is *positive*, and from  $P$  toward C is *negative*.

Let  $t$  = time of meeting in hours after 12 o'clock; then  $mt$  = distance from  $P$  to the point of meeting.

Since A traveled at the rate of  $m$  miles an hour, and B  $n$  miles an hour, we have

$$mt = \text{the distance A traveled.}$$

$$\text{And} \quad nt = \text{ " " B "}$$

Again, since A and B were  $d$  miles apart at 12 o'clock,

$$mt - nt = d$$

Factoring and dividing we have the

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\* Originally proposed by Clairaut, an eminent French mathematician, born in 1713.

$$\text{FORMULA.} \quad t = \frac{d}{m - n}.$$

The problem may now be discussed in relation to the time  $t$ , and the distance  $mt$ , the two unknown elements.

### I. Suppose $m > n$ .

Upon this supposition the values of  $t$  and  $mt$  will both be positive; because their denominator  $m - n$  is positive. Now since  $t$  is *positive*, it is evident the two couriers came together *after* 12 o'clock; and as  $mt$  is *positive*, the point of meeting was somewhere on the right of  $P$ .

These conclusions agree with each other, and correspond to the conditions of the problem. For, the supposition that  $m > n$  implies that  $A$  was traveling faster than  $B$ .  $A$  would therefore gain upon  $B$ , and overtake him some time after 12 o'clock, and at a point in the direction of  $Q$ .

Let  $d = 24$  miles,  $m = 8$  miles, and  $n = 6$  miles.

By the formula,  $t = \frac{d}{m - n} = \frac{24}{8 - 6} = 12$  hours.

$$mt = 8 \times 12 = 96 \text{ miles } A \text{ traveled.}$$

$$nt = 6 \times 12 = 72 \quad " \quad B \quad "$$

Now,  $96 - 72 = 24$   $m$ . their distance apart at noon, as given above.

These values show that the couriers were together in 12 hours past noon, or at midnight, and at a point  $Q$ , 96 miles from  $P$  and 72 miles from  $d$ .

### II. Suppose $m < n$ .

Then in the formula, the denominator  $m - n$  is negative, therefore both  $t$  and  $mt$  are negative.

Hence, both  $t$  and  $mt$  must be taken in a sense contrary to that which they had in supposition (I), where they were positive; that is, the time the couriers were together was *before* 12 o'clock, and the place of meeting on the *left* of  $P$ .

This interpretation is also in accordance with the conditions of the problem under the present supposition. For, if  $m < n$ , then  $B$  was traveling faster than  $A$ ; and as  $B$  was in advance of  $A$  at 12 o'clock, he must have passed  $A$  before that time, somewhere on the left of  $P$ , in the direction of  $C$ .

Let  $d = 24$  miles,  $m = 5$  miles, and  $n = 8$  miles.

By the formula,  $t = \frac{d}{m - n} = \frac{24}{5 - 8} = -8$  hours.

$$\text{And} \quad mt = 5 \times -8 = -40 \text{ miles } A \text{ traveled}$$

$$nt = 8 \times -8 = -64 \quad " \quad B \quad "$$

These values show that the couriers were together 8 hours before noon, or at 4 o'clock A. M., and at a point C, 40 miles from  $P$  and 64 miles from  $d$ .

### III. Suppose $m = n$ .

Upon this supposition we have  $m - n = 0$ , and

$$t = \frac{d}{0} = \infty, \text{ also } mt = \frac{md}{0} = \infty.$$

According to these results,  $t$  the time to elapse before the couriers are together, is infinity (Art. 434): consequently they can never be together. In like manner  $mt$ , the distance from  $P$  of the supposed point of meeting, is infinity; hence, there can be no such point.

This interpretation agrees with the supposed conditions of the problem. For, at 12 o'clock the two couriers were  $d$  miles apart, and if  $m = n$  they were traveling at equal rates, and therefore, could never meet.

### IV. Suppose $d = 0$ , and $m$ either greater or less than $n$ .

We then have  $t = \frac{0}{m - n} = 0$ , and  $mt = 0$ .

That is, both the time and distance are *nothing*. These results show that the couriers were together at 12 o'clock at the point  $P$ , and at no other time or place.

This interpretation is confirmed by the conditions of the problem. For, if  $d = 0$ , then at 12 o'clock B must have been with A at the point  $P$ . And if  $m$  is greater than  $n$ , or  $m$  is less than  $n$ , the couriers were traveling at different rates, and must either approach or recede from each other at all times, except at the moment of passing; therefore they can be together only at a single point.

### V. Suppose $d = 0$ , and $m = n$ .

Then we have  $t = \frac{0}{0}$ , and  $mt = \frac{0}{0}$ .

These results must be interpreted to mean that the time and the distance may be anything whatever, and that the couriers must be together at all times, and at any distance from  $P$ .

This conclusion also corresponds to the conditions of the problem. For, if  $d = 0$ , the couriers were together at 12 o'clock, and if  $m = n$ , they were traveling at equal rates, and therefore would never part.

## IMAGINARY QUANTITIES.

**510.** An *Imaginary Quantity* is an indicated *even* root of a *negative* quantity; as,  $\sqrt{-1}$ ,  $\sqrt{-a}$ ,  $\sqrt[4]{-7}$ .

NOTES.—1. Imaginary quantities are a species of *radicals*, and are called *imaginary*, because they denote operations which it is impossible to perform. (Art. 294.)

2. Though the operations indicated are in themselves *impossible*, these imaginary expressions are often useful in mathematical analyses, and when subjected to certain modifications, lead to important results.

**511.** Imaginary quantities are *added* and *subtracted* like other radicals. (Arts. 310, 311.)

But to *multiply* and *divide* them, some modifications in the rules of radicals are required. (Arts. 312, 313.)

**512.** To Prepare an *Imaginary Quantity* for Multiplication and Division.

RULE.—*Resolve the given quantity into two factors, one of which is a real quantity, and the other the imaginary expression  $\sqrt{-1}$ .*

NOTES.—1. This modification is based upon the principle that any *negative* quantity may be regarded as the product of two quantities, one of which is  $-1$ . Thus,  $-a = a \times -1$ ;  $-b^2 = b^2 \times -1$ .

2. The real factor is often called the coefficient of the imaginary expression,  $\sqrt{-1}$ .

1. Multiply  $\sqrt{-a}$  by  $\sqrt{-b}$ .

SOLUTION.  $\sqrt{-a} = \sqrt{a} \times \sqrt{-1}$ , and  $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$ .  
Now  $\sqrt{a} \times \sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab} \times -1 = -\sqrt{ab}$ , Ans.

2. Multiply  $+\sqrt{-x}$  by  $-\sqrt{-y}$ .

3. Multiply  $\sqrt{-9}$  by  $\sqrt{-4}$ .

510. What are imaginary quantities? 511. How added and subtracted?  
512. How prepare them for multiplication and division?

**513.** It will be seen from the preceding examples:

*First.* That the product of two imaginary quantities is a real quantity.

*Second.* That the sign before the product is the *opposite* of that required by the common rule for signs. (Art. 92.)

For, while the sign to be prefixed to an even root is ambiguous, this ambiguity is removed when we know whether the quantity whose root is to be taken has been produced from positive or negative quantities. (Art. 293.)

4. Multiply  $\sqrt{-2}$  by  $\sqrt{18}$ ,

5. Multiply  $\sqrt{-x}$  by  $\sqrt{y}$ .

NOTE.—1. From these examples it will be seen that the product of a real quantity and an imaginary expression, is itself imaginary.

6. Divide  $\sqrt{-x}$  by  $\sqrt{-y}$ .

$$\text{SOLUTION. } \frac{\sqrt{-x}}{\sqrt{-y}} = \frac{\sqrt{x} \times \sqrt{-1}}{\sqrt{y} \times \sqrt{-1}} = \sqrt{\frac{x}{y}}, \text{ Ans.}$$

7. Divide  $\sqrt{-x}$  by  $\sqrt{-x}$ ,

NOTE.—2. Hence, the *quotient* of one imaginary quantity divided by another, is a *real* quantity; and the sign before the radical is the same as that prescribed by the rule. (Art. 92.)

8. Divide  $\sqrt{-x}$  by  $\sqrt{y}$ ,

9. Divide  $\sqrt{x}$  by  $\sqrt{-y}$ ,

NOTE.—3. Hence, the *quotient* of an imaginary quantity divided by a real one, is itself imaginary, and *vice versa*.

10. Divide  $10\sqrt{-14}$  by  $2\sqrt{-7}$ ,

11. Divide  $c\sqrt{-1}$  by  $d\sqrt{-1}$ ,

**514.** The development of the different powers of  $\sqrt{-1}$ .

$$12. (\sqrt{-1})^2 = -1.$$

$$15. (\sqrt{-1})^5 = +\sqrt{-1}.$$

$$13. (\sqrt{-1})^3 = -\sqrt{-1}.$$

$$16. (\sqrt{-1})^6 = -1.$$

$$14. (\sqrt{-1})^4 = +1.$$

$$17. (\sqrt{-1})^7 = -\sqrt{-1}.$$

Hence, the *even* powers are alternately  $-1$  and  $+1$ , and the *odd* powers  $-\sqrt{-1}$  and  $+\sqrt{-1}$ .

# INDETERMINATE PROBLEMS.

**515.** An *Indeterminate Problem* is one which does not admit of a definite answer. (Art. 220.)

NOTE.—Among the more common indeterminate problems, are

1st. Those whose conditions are satisfied by different values of the same unknown quantity. (Art. 220.)

2d. Those which produce identical equations. (Art. 200.)

3d. Those which have a less number of independent simultaneous equations than there are unknown quantities to be determined.

4th. Those whose conditions are inconsistent with each other.

1. Given the equation  $x + y = 9$ , to find the value of  $x$ .

SOLUTION.—Transposing,  $x = 9 - y$ , Ans. This result can be verified by assigning any values to  $x$  or  $y$ .

2. What number is that,  $\frac{1}{4}$  of which minus 1 half of itself is equal to its 12th part plus its sixth part?

Let  $x =$  the number,

Then  $\frac{3x}{4} - \frac{x}{2} = \frac{x}{12} + \frac{x}{6}$

Clearing of fractions, etc.,  $9x = 9x$

Transposing and factoring,  $(9-9)x = 0$

$$\therefore x = \frac{0}{0}$$

# IMPOSSIBLE PROBLEMS.

**516.** An *Impossible Problem* is one, the conditions of which are *contradictory* or *impossible*.

1. Given  $x + y = 10$ ,  $x - y = 2$ , and  $xy = 38$ .

OPERATION.

SOLUTION.—By combining equations (1) and (2), we find  $x = 6$  and  $y = 4$ . Again,  $x \times y = 6 \times 4 = 24$ . But the third condition requires the product of  $x$  and  $y$  to be 38, which is impossible.

$$\begin{array}{rcl} x + y & = & 10 \quad (1) \\ x - y & = & 2 \quad (2) \\ \hline 2x & = & 12 \\ \therefore x & = & 6 \\ y & = & 4 \end{array}$$

2. What number is that whose 5th part exceeds its 4th part by 15?
3. Divide 8 into two such parts that their product shall be 18.

## NEGATIVE SOLUTIONS.

**517.** A *Negative Solution* is one whose result is a *minus* quantity.

**518.** An *odd root* of a quantity has the same sign as the quantity. An *even root* of a *positive* quantity is either *positive* or *negative*, both being numerically the same. (Art. 293.)

But the results of problems in Simple Equations, it is understood, are *positive*; when otherwise it is presumed there is an *error* in the data, which being corrected, the result will be positive.

1. A school-room is 30 feet long and 20 feet wide. How many feet must be added to its width that the room may contain 510 square feet?

SOLUTION.—Let  $x$  = the number of feet,  
 Then  $(20 + x) 30$  = area.  
 By conditions,  $600 + 30x$  = 510  
 Transposing,  $30x$  = -90  
 $\therefore x$  = -3 ft., *Ans.*

NOTES.—1. It will be observed that this is a problem in Simple Equations. The steps in the solution are legitimate and the result satisfies the conditions of the problem algebraically, but not arithmetically. Hence, the *negative result* indicates some *mistake* or *inconsistency* in the conditions of the problem.

If we subtract 3 ft. from its width, the result will be, a *positive* quantity.

2. Were it asked how much must be added to the width that the room may contain 690 square feet, the result would be +3 feet.

3. In such cases, by changing some of the data, a similar problem may be easily found whose conditions are consistent with a possible result.

2. What number must be subtracted from 5 that the remainder may be 8?

SOLUTION.—Let

$x$  = the number.

Then

$$5 - x = 8$$

Transposing,

$$x = -3, \text{ Ans.}$$

3. A man at the time of his marriage was 36 years old and his wife 20 years. How many years before he was twice as old as his wife? Ans. — 4 years.

## HORNER'S METHOD OF APPROXIMATION.\*

**519.** This method consists in transforming the given equation into another whose root shall be less than that of the given equation by the first figure of the root, and repeating the operation till the desired approximation is found.

The process may be illustrated in the following manner:

Let it be required to find the approximate value of  $x$  in the general equation,

$$Ax^3 + Bx^2 + Cx = D. \quad (1)$$

Having found the first figure of the root by trial, let it be denoted by  $a$ , the second figure by  $b$ , the third by  $c$ , and so on.

Substituting  $a$  for  $x$  in equation (1), we have,

$$Aa^3 + Ba^2 + Ca = D, \text{ nearly.}$$

Factoring and dividing,

$$a = \frac{D}{0 + Ba + Aa^2} \quad (2)$$

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\* So called from the name of its author, an English mathematician, who communicated it to the Royal Society in 1819.

By putting  $y$  for the sum of all the figures of the root except the first, we have  $x = a + y$ , and substituting this value for  $x$  in equation (1), we have,

$$A(a+y)^2 + B(a+y)^2 + C(a+y) = D;$$

$$\text{or } A(a^2 + 2ay + y^2) + B(a^2 + 2ay + y^2) + C(a+y) = D.$$

$$\text{or } Aa^2 + 3Aa^2y + 3Aay^2 + Ay^3 + Ba^2 + 2Bay + By^2 + Ca + Cy = D.$$

Factoring and arranging the terms according to the powers of  $y$ , we obtain

$$Ay^3 + (B + 3Aa)y^2 + (C + 2Ba + 3Aa^2)y = D - (Ca + Ba^2 + Aa^3). \quad (3)$$

To simplify this equation, let us denote the coefficient of  $y^2$  by  $B'$ , that of  $y$  by  $C'$ , and the second member by  $D'$ ; then,

$$Ay^3 + B'y^2 + C'y = D'. \quad (4)$$

It will be seen that equation (4) has the same form as (1). It is the first transformed equation, and its root is less by  $a$  than the root of equation (1).

By repeating the operation, a second transformed equation may be obtained. Denoting the second figure in the root by  $b$ , and reducing as before, we find,

$$b = \frac{D'}{C' + B'b + Ab^2}. \quad (5)$$

Putting  $z$  for the sum of all the remaining figures in the root, we have  $y = b + z$ ; and substituting this value in equation (4), we obtain a new equation of the same general form, which may be written,

$$Az^3 + B''z^2 + C''z = D''. \quad (6)$$

This process should be continued till the desired accuracy is attained. The first figure of the root is found by trial, the second figure from equation (5), and the remaining figures can be found from similar equations.

But it may be observed that the second member of equation (5) involves the quantity  $b$ , whose value is sought. That is, the value of  $b$  is given in terms of  $b$ , and that of  $c$  would be given in terms of  $c$ , and so on. For this reason, equations such as (5) might appear at first sight to be of little use in practice. This, however, is not the case; for after the root has been found to several decimal places, the value of the second and third terms, as  $B'b + Ab^3$  and  $B''c + Ac^3$  in the denominators, will be very small compared with  $C'$  and  $C''$ , conse-

quently as  $b$  is very nearly equal to  $D'$  divided by  $C'$ , they may be neglected. Therefore the successive figures in the root may be approximately found by dividing  $D'$  by  $C'$ ,  $D''$  by  $C''$ , and so on, regarding  $C'$ ,  $C''$ , etc., as approximate divisors.

In transforming equation (1) into (4), the second member  $D'$  and the coefficients  $C'$  and  $B'$  of the transformed equation may be thus obtained.

Multiplying the first coefficient  $A$  by  $a$ , the first figure of the root, and adding the product to  $B$ , the second coefficient, we have,

$$B + Aa \quad (7)$$

Again, multiplying this expression by  $a$ , and adding the product to  $C$ , the third coefficient, we have,

$$C + Ba + Aa^2. \quad (8)$$

Finally, multiplying these terms by  $a$ , and subtracting the product from  $D$ , we have

$$D - (Ca + Ba^2 + Aa^3) = D',$$

which is the same as the expression for  $D'$  in equation (4).

Now to obtain  $C'$ , we return to the first coefficient, multiply it by  $a$ , add the product to expression (7), and thus have the sum

$$B + 2Aa, \quad (9)$$

which we multiply by  $a$ , and adding the product to expression (8) obtain,

$$C + 2Ba + 3Aa^2 = C',$$

which is the desired coefficient of  $y$  in equation (4).

Finally, to obtain  $B'$ , we multiply the first coefficient by  $a$ , and add the product to expression (9), and thus obtain,

$$B + 3Aa = B'.$$

In this way the coefficients of the first transformed equation are discovered; and by a similar process the coefficients of the second, third, and of all subsequent transformed equations may be found.

**520.** This method of approximation is applicable to equations of every degree. For the solution of cubic equations, it may be summed up in the following

**RULE.**—I. *Detach the coefficients of the given equation, and denote them by  $A$ ,  $B$ ,  $C$ , and the second member by  $D$ . Find the first figure of the root by trial, and represent it by  $a$ .*

*Multiply A by a, and add the product to B. Multiply the sum by a and add the product to C. Multiply this sum by a and subtract the product from D. The remainder is the first dividend, or D'.*

*II. Multiply A by a and add the product to the last sum under B. Multiply this sum by a and add the product to the last sum under C. The result thus obtained is the first divisor, or C'.*

*III. Multiply A by a and add the product to the last sum under B. The result is the second coefficient, or B'.*

*IV. Divide the first dividend by the first divisor. The quotient is the second figure of the root, or b.*

*V. Proceed in like manner to find the subsequent figures of the root.*

**NOTE.**—I. In finding the second figure of the root, some allowance should be made for the terms in the divisor which are disregarded; otherwise the quotient will furnish a result too large to be subtracted from D'.

### EXAMPLES.

1. Given  $x^3 + 2x^2 + 3x = 24$ , to find  $x$ .

#### SOLUTION.

A	B	C	D	a b c
1	+2	+3	= 24	$x = (2.08, Ans$
	<u>2</u>	<u>8</u>	<u>22</u>	
	4	11	2 = D'	
	<u>2</u>	<u>12</u>	<u>1.891712</u>	
	6	23 = C'	.108288 = D''	
	<u>2</u>	<u>.6464</u>		
	8 = B'	23.6464		
	<u>.08</u>	<u>.6528</u>		
	8.08	24.2992 = C''		
	<u>.08</u>			
	8.16			
	<u>.08</u>			
	9.24 = B''			

NOTE.—2. In the following example, the last figures of the root are found by the contracted method of division of decimals, an expedient which may always be used to advantage after a few places of decimals have been obtained. (See Higher Arithmetic.)

2. Given  $x^3 + 12x^2 - 18x = 216$ , to find  $x$ .

SOLUTION.

<i>A</i>	<i>B</i>	<i>O</i>	<i>D</i>	<i>a b c</i>
1	+12	-18	= 216	( 4.24264 +.
	<u>4</u>	<u>+64</u>	<u>184</u>	
	16	+46	32 = <i>D'</i>	
	<u>4</u>	<u>80</u>	<u>26.168</u>	
	20	126 = <i>O'</i>	5.832 = <i>D''</i>	
	<u>4</u>	<u>4.8 4</u>	<u>5.468224</u>	
	24 = <i>B'</i>	130.8 4	.363776 = <i>D'''</i>	
	24.2	<u>4.8 8</u>	<u>275385</u>	
	24.4	135.7 2 = <i>O''</i>	88391	
	24.6 = <i>B''</i>	<u>.9 8 5 6</u>	<u>82615</u>	
	24.64	136.7 0 5 6	5776	
	24.68	<u>.9 8 7 2</u>	<u>5508</u>	
		137.6 9 2 8 = <i>O'''</i>		

$x = 4.24264 +$ , *Ans.*

3. Given  $x^3 + 3x^2 + 5x = 178$ , to find  $x$ .

$x = 4.5388$ , *Ans.*

4. Given  $5x^3 + 9x^2 - 7x = 2200$ , to find  $x$ .

$x = 7.1073536$ , *Ans.*

5. Given  $x^3 + x^2 + x = 100$ , to find  $x$ .

$x = 4.264429 +$ , *Ans.*

## TEST EXAMPLES FOR REVIEW.

1. Required the value of  
 $6a + 4a \times 5 + 8a \div 2 - 3a + 12a \times 4$
2. Required the value of  
 $(8x + 3x) 5 + 4x + 7 - (5x + 9x) \div 7$
3. Required the value of  
 $5ax - ab + 4cd - (2ax - 4ab + 2cd)$
4. Required the value of  
 $4bc + [3cd - (2xy - mn) 5 + 3bc]$
5. Show that subtracting a negative quantity is equivalent to adding a positive one.
6. Explain by an example why a positive quantity multiplied by a negative one produces a negative quantity?
7. Explain why a minus quantity multiplied by a minus quantity produces a positive quantity.
8. Given  $\frac{2x}{3} - (x + 8) = \frac{48}{9} + \frac{12}{7} - 17\frac{1}{2}$ , to find  $x$ .
9. Given  $\frac{4x^3}{5} \div \frac{x}{5} + 2x = \frac{36}{3} \times \frac{31}{2}$ , to find  $x$ .
10. Resolve  $3b^2c - 6b^2c^2 - c^2d$  into two factors.
11. Resolve  $3x^2y - 9x^2z - 18x^2yz$  into two factors.
12. Resolve  $a^{2n} - b^{2n}$  into two factors.
13. Resolve  $8a - 4$  into prime factors.
14. Resolve  $a^4 - 1$  into prime factors.
15. Divide 31 into two such parts that 5 times one of them shall exceed 9 times the other by 1.
16. Make an algebraic formula by which any two numbers may be found, their sum and difference being given.
17. Two sportsmen at Creedmoor shoot alternately at a target; A hits the bull's-eye 2 out of 3 shots, and B 3 out of 4 shots; both together hit it 34 times. How many shots did each fire?

18. Find two quantities the product of which is  $a$  and the quotient  $b$ .

19. Reduce  $\frac{a^2 - 1}{ab - b}$  to its lowest terms.

20. Reduce  $\frac{a + b}{a^2 - b^2}$  to its lowest terms.

21. Resolve  $9x^2y^2 + 12xyz + 4z^2$  into two factors.

22. Resolve  $9b^2 - 6bc + c^2$  into two factors.

23. Make a formula by which the width of a rectangular surface may be found, the area and length being given?

24. A square tract of land contains  $\frac{1}{4}$  as many acres as there are rods in the fence inclosing it. What is the length of the fence?

25. A student walked to the top of Mt. Washington at the rate of  $1\frac{1}{4}$  miles an hour, and returned the same day at the rate of  $4\frac{1}{4}$  miles an hour; the time occupied in traveling being 13 hours. How far did he walk?

26. Given  $b - \frac{1+x}{1-x} = 0$ , to find  $x$ .

27. Prove that the product of the sum and difference of two quantities, is equal to the difference of their squares.

28. Prove that the product of the sum of two quantities into a third quantity, is equal to the sum of their products.

29. Reduce  $\frac{(x^2 - y^2)(x + y)}{(x^2 + 2xy + y^2)(x - y)}$  to its lowest terms.

30. Reduce  $\frac{a^4 - b^4}{(a^2 - 2ab + b^2)(a^2 + b^2)}$  to its lowest terms.

31. Reduce  $\frac{1+a^2}{1-a^4} - \frac{1-a^2}{1+a^2}$  to a single fraction having the least common denominator.

32. Find a number to which if its fourth and fifth part be added, the sum will exceed its sixth part by 154.

33. Two persons had equal sums of money; the first spent \$30, the second \$40: the former then had twice as much as the latter. What sum did each have at first?

34. A French privateer discovers a ship 24 kilometers distant, sailing at the rate of 8 kilometers an hour, and pursues her at the rate of 12 kilometers an hour. How long will the chase last?

35. Given  $\frac{8x + 3y}{7} = 7$  and  $\frac{7y - 3x}{2} - y = 0$ , to find  $x$  and  $y$ .

36. Given  $x = \frac{y - 2}{7} + 5$  and  $4y - \frac{x + 10}{3} = 3$ , to find  $x$  and  $y$ .

37. Make a rule to find when any two bodies moving toward each other will meet, the distance between them and the rate each moves being given?

38. A steamer whose speed in still water is 12 miles an hour, descended a river whose velocity is 4 miles an hour, and was gone 8 hours. How far did she go in the trip?

39. Find a fraction from which if 6 be subtracted from both its terms it becomes  $\frac{1}{2}$ , and if 6 be added to both, it becomes  $\frac{3}{4}$ .

40. Required two numbers whose sum is to the less as 8 is to 3, and the difference of whose squares is 49.

41. Given  $10x + 6y = 76$ ,  $4y - 2z = 8$ , and  $6x + 8z = 88$ , to find  $x$ ,  $y$ , and  $z$ .

42. Given  $2x + 3y + z = 24$ ,  $3x + y + 2z = 26$ , and  $x + 2y + 3z = 34$ , to find  $x$ ,  $y$ , and  $z$ .

43. Three persons, A, B, and C, counting their money, found they had \$180. B said if his money were taken from the sum of the other two, the remainder would be \$60; C said if his were taken from the sum of the other two, the remainder would be  $\frac{1}{4}$  of his money. How much money had each?

44. The fore-wheel of a steam-engine makes 40 revolutions more than the hind-wheel in going 240 meters, and the circumference of the latter is 3 meters greater than that of the former. What is the circumference of each?

45. A man has two cubical piles of wood; the side of one

is two feet longer than the side of the other, and the difference of their contents is 488 cubic feet. Required the side of each.

46. Required a formula by which the height of a rectangular solid may be found, the contents and base being given.

47. Divide 126 into two such parts that one shall be a multiple of 7, the other a multiple of 11.

48. A tailor paid \$120 for French cloths; if he had bought 8 meters less for the same money, each meter would have cost 50 cents more. How many meters did he buy?

49. A shopkeeper paid \$175 for 89 meters of silk. At what must he sell it a meter to make 25 per cent?

50. Make a formula to find the commercial discount, the marked price and the rate of discount being given.

51. A man pays \$100 more for his carriage than for his horse, and the price of the former is to that of the latter as the price of the latter is to 50. What is the price of each?

52. Make a formula to find at what time the hour and minute hands of a watch are together between any two consecutive hours?

53. A father bequeathed 165 hektars of land to his two sons, so that the elder had 35 hektars more than the younger. How many hektars did each receive?

54. What number is that, the triple of which exceeds 40 by as much as its half is less than 51?

55. A butcher buys 6 sheep and 7 lambs for \$71; and, at the same price, 4 sheep and 8 lambs for \$64. What was the price of each?

56. At a certain election, 1425 persons voted, and the successful candidate had a majority of 271 votes. How many voted for each?

57. A's age is double B's, and B's is three times C's; the sum of all their ages is 150. What is the age of each?

58. Reduce the  $\sqrt{243}$  to its simplest form.

59. Reduce  $\sqrt{y^2 + ay^2}$  to its simplest form.

60. Reduce  $x^{\frac{1}{2}}$  and  $y^{\frac{1}{3}}$  to the common index  $\frac{1}{6}$ .
61. Reduce  $3(a - b)$  to the form of the cube root.
62. A farmer sold 13 bushels of corn at a certain price; and afterward 17 bushels at the same rate, when he received \$3.60 more than at the first sale. What was the price per bushel?
63. A sold two stoves. On the first he lost \$8 more than on the second; and his whole loss was \$2 less than triple the amount lost on the second. How much did he lose on each?
64. A number of men had done  $\frac{1}{4}$  of a piece of work in 6 days, when 12 more men were added, and the job was completed in 10 days. How many men were at first employed?
65. A company discharged their bill at a hotel by paying \$8 each; if there had been 4 more to share in the payment, they would only have paid \$7 apiece. How many were there in the party?
66. In one factory 8 women and 6 boys work for \$72 a week; and in another, at the same rates, 6 women and 11 boys work for \$80 a week. How much does each receive per week?
67. What factor can be removed from  $\sqrt{153x}$ ?
68. Given  $\sqrt{x + 12} = \sqrt{a + 12}$ , to find  $x$ .
69. Given  $\frac{\sqrt{y}}{y} = \frac{y - ay}{\sqrt{y}}$ , to find  $y$ .
70. Given  $\sqrt{x^2 - 4ab} = a - b$ , to find  $x$ .
71. From a cask of molasses  $\frac{1}{3}$  of which had leaked out, 40 liters were drawn, leaving the cask half full. How many liters did it hold?
72. Make a formula to find the per cent commission a factor receives, the amount invested and the commission being given.
73. Divide 20 into two parts, the squares of which shall be in the ratio of 4 to 9.

74. After paying out  $\frac{1}{3}$  of my money and then  $\frac{1}{3}$  of the remainder, I had \$140 left. How much had I at first?

75. If 1 be added to both terms of a fraction, its value will be  $\frac{3}{4}$ ; and if the denominator be doubled and then increased by 2, the value of the fraction will be  $\frac{1}{3}$ . Required the fraction.

76. Tiffany & Co. sold a gold watch for \$171, and the per cent gained was equal to the number of dollars the watch cost. Required the cost of the watch.

77. Two Chinamen receive the same sum for their labor; but if one had received \$15 more and the other \$9 less, then one would have had 3 times as much as the other. What did each receive?

78. A drover bought a flock of sheep for \$120, and if he had bought 6 more for the same sum, the price per head would have been \$1 less. Required the number of sheep and the price of each.

79. A certain number which has two digits is equal to 9 times the sum of its digits, and if 63 be subtracted from the number, its digits will be inverted. What is the number?

80. Two river-boatmen at the distance of 150 miles apart, start to meet each other; one rows 3 miles while the other rows 7. How far does each go?

81. A and B buy farms, each paying \$2800. A pays \$5 an acre less than B, and so gets 10 acres more land. How many acres does each purchase?

82. Find a factor that will rationalize  $\sqrt{x} + \sqrt{7}$ .

83. Find a factor that will rationalize  $\sqrt{3x} - \sqrt{3y}$ .

84. Given  $\sqrt{b^2 + x} = \frac{d + 3}{\sqrt{(b^2 + x)}}$ , to find  $x$ .

85. The salaries of a mayor and his clerk amount to \$13200; the former receives 10 times as much as the latter. Required the pay of each.

86. What two numbers are those whose sum is to their

difference as 8 to 6, and whose difference is to their product as 1 to 36?

87. What two numbers are those whose product is 48, and the difference of their cubes is to the cube of their difference as 37 to 1?

88. Find the price of apples per dozen, when 2 less for 12 cents raises the price 1 cent per dozen.

89. Two pedestrians set out at the same time from Troy and New York, whose distance apart is 150 miles; one goes at the rate of 24 m. in 3 days, and the other 14 m. in 2 days. When will they meet?

90. The income of A and B for one month was \$1876, and B's income was 3 times A's. Required that of each?

91. A farmer bought a cow and a horse for \$250, paying 4 times as much for the horse as for the cow. Find the cost of each.

92. A man rode 24 miles, going at a certain rate; he then walked back at the rate of 3 miles per hour and consumed 12 hours in making the trip. At what rate did he ride?

93. It costs \$6000 to furnish a church, or \$1 for every square foot in its floor. How large is the building, provided the perimeter be 320 feet?

94. Find 5 arithmetical means between 3 and 31.

95. Find the sum of 50 terms of the series  $\frac{1}{2}$ , 1,  $1\frac{1}{2}$ , 2,  $2\frac{1}{2}$ , 3,  $3\frac{1}{2}$ , 4,  $4\frac{1}{2}$ , etc.

96. A dealer bought a box of shoes for \$100. He sold all but 5 pair for \$135, at a profit of \$1 a pair. How many pair were there in the box?

97. Two numbers are to each other as 7 to 9, and the difference of their squares is 128. Required the numbers.

98. In a pile of scantling there are 2400 pieces, and the number in the length of the pile exceeds that in the height by 43: required the number in its height and length.

99. Bertha is  $\frac{1}{3}$  as old as her mother, but in 20 years she will be  $\frac{2}{3}$  as old. What is the age of each?

100. Fifteen persons engage a car for an excursion; but

before starting 3 of the company decline going, by which the expense of each is increased by \$1.75. What do they pay for the car?

101. When the hour and minute hands of a clock are together between 8 and 9 o'clock, what is the time of day?

102. A and B wrote a book of 570 pages; if A had written 3 times and B 5 times as much as each actually did write, they would together have written 2350 pages. How many pages did each write?

103. A man and his wife drink a pound of tea in 12 days. When the man is absent, it lasts the woman 30 days. How long will it last the man alone?

104. Find the time in which any sum of money will double itself at 7 per cent simple interest.

105. A purse contains a certain sum, in the proportion of \$3 of gold to \$2 of silver; if \$24 in gold be added, there will then be \$7 of gold for every \$2 of silver. Required the sum in the purse.

106. A and B in partnership gain \$3000. A owns  $\frac{2}{3}$  of the stock, lacking \$200, and gains \$1600. Required the whole stock and each man's share of it.

107. In the choice of a Chief Magistrate, 369 electoral votes were cast for two men. The successful candidate received a majority of one over his rival: how many votes were cast for each?

108. Two ladies can do a piece of sewing in 16 days; after working together 4 days, one leaves, and the other finishes the work alone in 36 days more. How long would it take each to do the work?

109. If a certain number be divided by the product of its two digits, the quotient is  $2\frac{1}{2}$ ; and if 9 be added to the number, the digits will be inverted: what is the number?

110. Find 4 geometrical means between 2 and 486.

111. A trader bought a number of hats for \$80; if he had bought 4 more for the same amount, he would have paid \$1 less for each: how many did he buy?

112. If the first term of a geometrical series is 2, the ratio 5, and the number of terms 12, what is the last term?

113. A tree 90 feet high, in falling broke into three unequal parts; the longest piece was 5 times the shortest, and the other was 3 times the shortest: find the length of each piece.

114. The sum of 3 numbers is 219; the first equals twice the second increased by 11, and the second equals  $\frac{2}{3}$  of the remainder of the third diminished by 19: required the numbers.

115. Required 3 numbers in geometrical progression, such that their sum shall be 14 and the sum of their squares 84.

116. A pound of coffee lasts a man and wife 3 weeks, and the man alone 4 weeks: how long will it last the wife?

117. Two purses contain together \$300. If you take \$30 from the first and put into the second, each will then contain the same amount: required the sum in each purse.

118. A clothier sells a piece of cloth for \$39 and in so doing gains a per cent equal to the cost. What did he pay for it?

119. A settler buys 100 acres of land for \$2450; for a part of the farm he pays \$20 and for the other part \$30 an acre. How many acres were there in each part?

120. What is the sum of the geometrical series 2, 6, 18, 54, etc., to 15 terms?

121. There are 300 pine and hemlock logs in a mill-pond, and the square of the number of pines is to the square of the number of hemlocks as 25 to 49: required the number of each kind.

122. A ship of war, on entering a foreign port, had sufficient bread to last 10 weeks, allowing each man 2 kilograms a week. But 150 of the crew deserted the first night, and it was found that each man could now receive  $3\frac{1}{2}$  kilograms a week for the remainder of the cruise. What was the original number of men?

## APPENDIX.

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### 521. To Extract the Cube Root of Polynomials.\*

1. Required the cube root of  $a^6 + 3a^5 - 3a^4 - 11a^3 + 6a^2 + 12a - 8$ .

OPERATION.

$a^6 + 3a^5 - 3a^4 - 11a^3 + 6a^2 + 12a - 8$  (  $a^3 + a - 2$ , Roots  
 $a^3$ , the first subtrahend.

1st Trial Divisor,  $3a^4$  )  $3a^5 - 3a^4 - 11a^3$ , etc., first remainder.

Com. D.,  $3a^4 + 3a^3 + a^2$  )  $3a^5 + 3a^4 + a^3$

2d Tr. D.,  $3a^4 + 6a^3 + 3a^2$  )  $-6a^4 - 12a^3 + 6a^2 + 12a - 8$ , 2d remainder.

Complete Divisor,

$3a^4 + 6a^3 - 3a^2 - 6a + 4$  )  $-6a^4 - 12a^3 + 6a^2 + 12a - 8$ . Hence, the

**RULE.**—I. *Arrange the terms according to the powers of one of the letters, take the cube root of the first term for the first term of the root, and subtract its cube from the given polynomial.*

II. *Divide the first term of the remainder by three times the square of the first term of the root as a trial divisor, and the quotient will be the next term of the root.*

III. *Complete the divisor by adding to it three times the product of the first term by the second, also the square of the second. Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.*

IV. *If there are more than two terms in the root, for the second trial divisor, take three times the square of the part of the root already found, and completing the divisor as before, continue the operation until the root of all the terms is found. (See Key)*

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\* For Horner's Method of Approximation, see p. 269.

2. Required the cube root of  $a^3 + 3a^2b + 3ab^2 + b^3$ .
3. Find the cube root of  $x^3 + 6x^2 + 12x + 8$ .
4. Find the cube root of  $x^3 - 6x^2y + 12xy^2 - 8y^3$ .
5. What is the cube root of  $8a^3 - 48a^2 + 96a - 64$ .
6. Find the cube root of  $27a^3 - 54a^2x + 36ax^2 - 8x^3$ .
7. What is the cube root of  $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$ .
8. The cube root of  $x^9 - 3x^8 + 8x^6 - 6x^5 - 6x^4 + 8x^3 - 3x + 1$ .

**522. Factor the following *Polynomials*.\***

- |                              |                                  |
|------------------------------|----------------------------------|
| 1. $x^2 - 9x + 20$ .         | 7. $x^3 + y^3z + ymz$ .          |
| 2. $a^2 + 7a - 18$ .         | 8. $12a^2x - 8a^2y + 4az$ .      |
| 3. $a^2 - 13a + 40$ .        | 9. $a^3 - 3a^2x + 3ax^2 - x^3$ . |
| 4. $2abc^2 - 14abc - 60ab$ . | 10. $1 - a^4$ .                  |
| 5. $x^2y^2 - 2xy + 1$ .      | 11. $1 + 8a^3$ .                 |
| 6. $8x^3 - 32y^2$ .          | 12. $a^6 - b^4x^2$ .             |

**523. Find the *g. c. d.* of the following *Polynomials*:**

1.  $4a^2 - 4ax - 15x^2$  and  $6a^2 + 7ax - 3x^2$ .
2.  $4ax^2y^4z^5$ ,  $12x^5z^6$ , and  $16a^3x^3z^2$ .
3.  $16x^2 - y^2$ , and  $16x^2 - 8xy + y^2$ .
4.  $6a^2 + 11ax + 3x^2$ , and  $6a^2 + 7ax - 3x^2$ .
5.  $a^4 - b^4$ , and  $a^5 - b^2a^3$ .
6.  $x^3 - a^3$ , and  $x^4 - a^4$ .

**524. Required the *l. c. m.* of the following *Polynomials*.**

1.  $6a^2 - 4a$ ,  $4a^3 + 2a$ , and  $6a^2 + 4a$ .
2.  $4(1 + a^2)$ ,  $8(1 - a)$ ,  $4(1 - a^2)$ , and  $8(1 + a)$ .
3.  $a^2 - 2a + 1$ ,  $a^4 - 1$ , and  $a^2 + 2a + 1$ .
4.  $12(ab^2 - b^3)$ ,  $4(a^2 + ab)$ , and  $18(a^2 - b^2)$ .
5.  $4a^2 - 1$ ,  $2a - 1$ , and  $4a^2 + 1$ .
6.  $4(1 + a^2)$ ,  $8(1 + a)$ ,  $4(1 - a^2)$ , and  $8(1 - a)$ .

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\* The following problems are classified and may be studied in connection with the subjects to which they refer, or be omitted till the other parts of the book are finished, at the option of the teacher.

**525. Unite the following Fractions:**

1.  $5x + \frac{x-2}{3} + 4x - \frac{3x-3}{5x}.$

2.  $\frac{1}{a^2-b^2} + \frac{2ab}{a^4-b^4}.$

3.  $3 + \frac{2a}{x} + 5 - \frac{3a-2x}{x} + 7 + \frac{x-a}{a}.$

4.  $\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a+ac}{ac}.$

5. From  $a + 3h - \frac{d-b}{2}$  take  $3a - h + \frac{d+b}{3}.$

6. From  $5x + \frac{x}{b}$  take  $2x - \frac{x-b}{c}.$

7. From  $a + x + \frac{x}{x^2-y^2}$  take  $a - x + \frac{1}{x+y}.$

8. From  $\frac{a-1}{a}$  take  $\frac{a}{a-1}.$

9. From  $\frac{1}{1-x}$  take  $\frac{2}{1-x^2}.$

**526. Multiply the following Polynomials:**

1.  $1 - \frac{a-b}{a+b} \times 2 + \frac{2b}{a-b}.$  5.  $\frac{a^2}{x+y} \times \frac{x^2-y^2}{ab}.$

2.  $\frac{4a}{3x} + \frac{3x}{2b} \times \frac{2b}{3x} + \frac{3x}{4a}.$  6.  $a^2 - 1 \times \frac{5b}{a-1}.$

3.  $x^2 - 2xy + y^2 \times \frac{x+y}{x-y}.$  7.  $\frac{15a-30}{2a} \times \frac{3a^2}{5a-10}.$

4.  $\frac{7b}{2a^2-8a^4} \times \frac{2a^2-4a^3}{21b}.$  8.  $y - \frac{xy}{x+y} \times y + \frac{xy}{x-y}.$

**527. Divide the following Quantities:**

1.  $a + \frac{2a}{a-3} \div a - \frac{2a}{a-3}.$  4.  $\frac{3y}{2y-2} \div \frac{2y}{y-1}.$

2.  $\frac{1}{x} + \frac{1}{xy^2} \div y - 1 + \frac{1}{y}.$  5.  $\frac{ab+b^2}{a^3-b^3} \div \frac{b}{a-b}.$

3.  $1 + \frac{1}{x} \div 1 - \frac{1}{x^2}.$  6.  $\left(\frac{1}{1+a} + \frac{a}{1-a}\right) \div \frac{(1+a^2)^2}{(1-a^2)^2}.$

**528. Simplify the following Fractions:**

$$7. \frac{\frac{3a}{2a-2}}{\frac{2a}{a-1}}$$

$$8. \frac{x + \frac{x}{3}}{y + \frac{y}{4}}$$

$$9. \frac{\frac{x^2}{x^2-y^2} - 1}{\frac{y^2}{x^2-y^2} + 1}$$

$$10. \frac{\frac{1}{a} + \frac{1}{ab^3}}{b - 1 + \frac{1}{b}}$$

**529. Solve the following Equations:**

1. A house and barn cost \$850, and 5 times the price of the house was equal to 12 times the price of the barn. Required the price of each.

2. A, B, and C, together have 145 acres of land; A owns two-thirds and B three-fourths as much as C. How many acres has each?

3. From a cask  $\frac{1}{2}$  full of water, 21 liters leaked out, when  $\frac{1}{2}$  the water was left. Required the capacity of the cask.

$$4. 3x - \frac{x-4}{4} - 4 = \frac{5x+14}{3} - \frac{1}{12}.$$

$$5. \frac{3x+9}{2} = \frac{7x+5}{3} - \frac{16+4x}{5} + 6.$$

$$6. \frac{x+8}{4} - \frac{x-6}{3} + x = x + 2.$$

$$7. x - 2 = x + \frac{x+8}{4} - \frac{x-6}{3}.$$

$$8. \frac{2x+1}{3} - \left( \frac{-x-3}{4} \right) = x.$$

9. The hour and minute hands of a watch are together at 12 M. At what time between the hours of 7 and 8 P. M. will they again be in conjunction?

10. A merchant supported himself 3 yrs. for £50 a year, and at the end of each year added to that part of his stock which was not thus expended, a sum equal to  $\frac{1}{3}$  of this part. At the end of the third year his original stock was doubled. What was that stock?

**530. Solve the following *Simultaneous Equations*:**

1.  $\frac{4x + 6y}{3} = 26.$

3.  $\frac{3x}{4} - \frac{y}{2} = 9.$

$\frac{6x}{2} - \frac{6y}{3} = 0.$

$\frac{2x - 2y}{4} = 4.$

2.  $\frac{x}{3} + \frac{y}{2} = 8.$

4.  $\frac{x}{2} + \frac{y}{3} = b.$

$\frac{x}{2} + \frac{y}{3} = 7.$

$\frac{x}{3} + \frac{y}{4} = c.$

5. A man bought a horse, buggy, and harness for \$400; he paid 4 times as much for the horse as for the harness, and one-third as much for the harness as for the buggy; how much did he pay for each?

6. What number consisting of two figures is that to which, if the number formed by changing the place of the figures be *added*, the sum will be 121; but if *subtracted*, the remainder will be 9?

7.  $x + y - z = 0;$

10.  $xy = 600;$

$x + z - y = 2;$

$xz = 300;$

$y + z - x = 4.$

$yz = 200.$

8.  $\frac{x}{b} + \frac{y}{c} = 1;$

11.  $x + \frac{y}{4} - \frac{z}{8} = 19;$

$\frac{x}{b} + \frac{z}{d} = 1;$

$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 22;$

$\frac{y}{c} + \frac{z}{d} = 1.$

$\frac{x}{4} + y + \frac{z}{2} = 33.$

9.  $w + x + y = 6;$

12.  $w + 50 = x;$

$w + x + z = 9;$

$x + 120 = 3y;$

$w + y + z = 8;$

$y + 120 = 2z;$

$x + y + z = 7.$

$z + 195 = 3w.$

13. A's age added to 3 times B's and C's, is 470 yrs.; B's added to 4 times A's and C's, is 580 yrs.; and C's added to 5 times A's and B's, is 630 yrs. What age is each?

14. What 3 numbers are those whose sum is 59; half the difference of the first and second is 5, and half the difference of the first and third is 9?

**531.** Generalize the following Problems and translate the Formulas into Rules.

1. A dishonest clerk absconded, traveling 5 miles an hour; after 6 hours, a policeman pursued him, traveling 8 miles an hour. How long did it take the latter to overtake the former?

NOTE.—Substitute  $c$  for clerk's rate,  $p$  for policeman's rate,  $n$  for number of hours between starting, and  $x$  for the time required.

$$\text{FORMULA. } x = \frac{cn}{p - c}.$$

2. A can do a piece of work in 2 days, B in 5 days, and C in 10 days; how long will it take all working together to do it?

NOTE.—Let  $a$ ,  $b$ , and  $c$  represent the numbers. Then  $x = abc + (ab + ac + bc)$ .

$$\text{FORMULA. } x = \frac{abc}{ab + ac + bc}.$$

3. Divide \$4400 among A, B, and C, in proportion to the numbers 5, 7, and 10.

NOTE.—Put  $a$ ,  $b$ , and  $c$ , for the proportions,  $s$  for the sum of the proportions, and  $n$  for the number to be divided.

4. A father is now 9 times as old as his son; 9 years hence he will be only 3 times as old: what is the age of each?

**532.** Expand the following by the *Binomial Theorem*:

- |                    |                       |
|--------------------|-----------------------|
| 1. $(2a - 3b)^8$ . | 4. $(x^2 + y^2)^8$ .  |
| 2. $(3x + 2y)^4$ . | 5. $(a + a^{-1})^8$ . |
| 3. $(1 + 3a)^4$ .  | 6. $(a^2 - 2a)^6$ .   |

**533.** Find the Product of the following *Powers*:

- |   |                           |
|---|---------------------------|
| 1. $abx^{-6}$ by $x^8$ .                    | 3. $x^{-m}$ by $x^{-n}$ . |
| 2. $a^4b^{-6}x^{-8}$ by $a^{-2}b^3x^{-8}$ . | 4. $y^{-2}$ by $y^3$ .    |

**534.** Divide the following *Powers*:

- |  |                                   |
|--|-----------------------------------|
| 1. $6a^{-5}$ by $3a^{-2}$ .                | 3. $12x^{-5}$ by $4x^{-8}$ .      |
| 2. $8a^{-4}b^{-6}c^{-8}$ by $4a^2b^4c^5$ . | 4. $(a+x)^{-n}$ by $(a+x)^{-m}$ . |

**535.** Transfer Denominators to Numerators, thus forming entire Quantities.

1.  $\frac{x^2y}{z^3}$ .

3.  $\frac{3x^3}{2y^2z}$ .

2.  $\frac{a^3}{b^2c^2d}$ .

4.  $\frac{a^2x^3cm}{ax^5cm^4}$ .

**536.** Unite the following Radicals:

1.  $\sqrt{48} + \sqrt{27} + \sqrt{243}$ .

4.  $x\sqrt{25x^2c} + \sqrt{36x^4c}$ .

2.  $\sqrt{54x} - \sqrt{96x} + \sqrt{24x}$ .

5.  $\sqrt{80a^4x} - \sqrt{20a^2x}$ .

3.  $8\sqrt[3]{\frac{1}{3}b} + 2\sqrt[3]{\frac{1}{4}b}$ .

6.  $3\sqrt[3]{128x^3yz} - 4x\sqrt[3]{16yz}$ .

**537.** Find the *Product* of the following Radicals:

1.  $(a + y)^{\frac{1}{n}} \times (b + h)^{\frac{1}{n}}$ .

3.  $(x + y)^{\frac{1}{2}} \times (x + y)^{\frac{1}{2}}$ .

2.  $4 + 2\sqrt{2} \times 2 - \sqrt{2}$ .

4.  $3b\sqrt[3]{d + y} \times 4\sqrt{a}$ .

**538.** Dividing one Radical by Another.

1.  $(a^3h)^{\frac{1}{n}} \div (ax)^{\frac{1}{n}}$ .

4.  $(b + y)^{\frac{2}{3}} \div (b + y)^{\frac{1}{3}}$ .

2.  $24x\sqrt{ay} \div 6\sqrt{a}$ .

5.  $4a\sqrt{ab} \div 2\sqrt{ac}$ .

3.  $\sqrt{16a^3 - 12a^2x} \div 2a$ .

6.  $70\sqrt[3]{9} \div 7\sqrt[3]{18}$ .

**539.** Required the Factors which will *Rationalize* the following Radicals:

1.  $2\sqrt{a} + \sqrt{7}$ .

4.  $\sqrt{5} - \sqrt{x}$ .

2.  $x + \sqrt{y}$ .

5.  $4\sqrt{2x} - 5\sqrt{y}$ .

3. The denom. of  $\frac{b}{2\sqrt{3}}$ .

6. The d. of  $\frac{4}{\sqrt{3} + \sqrt{2} + 1}$ .

**540.** Solve the following *Radical Equations*:

1. Given  $\sqrt{x} + 1 = \sqrt{11 + x}$ , to find  $x$ .

2. Given  $\sqrt{x + 18} - \sqrt{5} = \sqrt{x - 7}$ , to find  $x$ .

3. Given  $\sqrt{x^2 - 11} = 5$ .

5.  $(13 + \sqrt{23 + y^2})^{\frac{1}{2}} = 5$ .

4.  $\frac{6}{\sqrt{3 + x}} = \sqrt{3 + x}$ .

6.  $2\sqrt{a} = \sqrt{x + 3a}$ .

**541. Solve the following Quadratics:**

$$1. \quad 5x - \frac{3x-3}{x-3} = 2x + \frac{3x-6}{2}.$$

$$2. \quad \frac{16}{x} - \frac{100-9x}{4x^2} = 3.$$

$$3. \quad \frac{x^6}{2} - \frac{x^8}{4} = -\frac{1}{32}.$$

$$4. \quad \frac{\sqrt{4x+2}}{4+\sqrt{x}} = \frac{4-\sqrt{x}}{\sqrt{x}}.$$

5. Find two numbers whose difference is 12, and the sum of their squares 1424.

6. Required two numbers whose sum is 6, and the sum of their cubes 72.

7. Divide the number 56 into two such parts, that their product shall be 640.

8. A and B started together for a place 150 miles distant. A's hourly progress was 3 miles more than B's, and he arrived at his journey's end 8 hrs. 20 min. before B. What was the hourly progress of each?

9. The difference of two numbers is 6; and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

10. The length added to the breadth of a rectangular room makes 42 feet, and the room contains 432 square feet. Required the length and breadth.

11. A says to B, the product of our years is 120; if I were 3 yrs. younger and you were 2 yrs. older, the product of our ages would still be 120? How old is each?

$$12. \quad \sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}.$$

$$13. \quad x + \sqrt{x+6} = 2 + 3\sqrt{x+6}.$$

14. A man bought 80 lbs. of pepper and 100 lbs. of ginger for £65, at such prices that he obtained 60 lbs. more of ginger for £20 than he did of pepper for £10. What did he pay per pound for each?

## COLLEGE EXAMINATION PROBLEMS.

- 542.** 1. Divide  $15x^3 - \frac{3ab}{5c}$  by  $x - \frac{a-b}{c}$ .
2. Divide  $a^4 - b^4$  by  $(a - b)$ .
3. Solve the equation  $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}$ .
4. Multiply  $3\sqrt{45} - 7\sqrt{5}$  by  $\sqrt{1\frac{1}{3}} + 2\sqrt{9\frac{1}{3}}$ .
5. Divide  $a^{\frac{1}{2}}b^{\frac{1}{3}}$  by  $a^{\frac{1}{3}}b^{\frac{1}{2}}$ . 6. Divide  $xy^{-1}$  by  $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ .
7. Given  $3x^2 + 2x - 9 = 76$ , to find  $x$ .
8. Given  $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{2}{3} = 8$ , to find  $x$ .
9. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.
10. Find the value of  $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \text{etc.}$  to infinity.
11. Find the third term of  $(a + b)^{15}$ .
12. Expand to four terms  $(1 + x^2)^{-\frac{1}{2}}$ .
- 543.** 1. Divide  $\frac{x+2y}{x+y} + \frac{x}{y}$  by  $\frac{x+2y}{y} - \frac{x}{x+y}$ .
2. Find the product of  $a^{\frac{1}{2}}, a^{\frac{1}{3}}, a^{\frac{1}{4}}$  and  $a^{-\frac{1}{12}}$ .
3. Solve the equation  $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$ .
4. Solve  $\frac{90}{x} - \frac{90}{x+1} - \frac{27}{x+2} = 0$ .
5. Solve  $\sqrt{x-1} = x-1$ .
6. Find the value of  $\frac{1}{3} + 1 + \frac{1}{3} + \text{etc.}$  to infinity.
7. Given  $x + \sqrt{x} : x - \sqrt{x} :: 3\sqrt{x} + 6 : 2\sqrt{x}$ , to find  $x$ .
8. Expand to four terms  $(a^2 + x)^{12}$ .
9. Expand to four terms  $(x^2 - y^2)^{-\frac{1}{2}}$ .
- 544.** 1. Find the *g. c. d.* and the *l. c. m.* of  $(243a^{10}b^5 + 1)$  and  $(81a^8b^4 - 1)$  by factoring.
2. Divide  $\frac{6\sqrt{b}}{25\sqrt[5]{a^3}}$  by  $\frac{20c\sqrt[4]{b^3}}{21ab\sqrt[3]{a^2}}$ .
3. Solve the equations  $2x - y = 21$ ,  $2x^2 + y^2 = 153$ .

4. A person buys cloth for \$90. If he had got two yards more for the same sum, the price would have been 50 cts. per yd. less. How much did he buy, and at what price?

5. Expand  $(a - b)^{12}$  by the binomial theorem.

6. Factor  $4a^2 - 9y^2$ .

7. Multiply  $3\sqrt{\frac{a}{3}}$  by  $2\sqrt{\frac{a}{6}}$ .

8. Given  $x + 2y = 7$  and  $2x + 3y = 12$ , to find  $x$  and  $y$ .

9. Reduce  $a\sqrt{48a^3d}$  and  $\sqrt{\frac{4}{3}}$  to their simplest forms.

10. Given  $\frac{x-5}{3} + \frac{x}{2} = 12 - \frac{x-10}{3}$ , to find  $x$ .

545. 1. From  $3x + \frac{x}{2b}$  subtract  $x - \frac{x-a}{c}$ .

2. Multiply together  $\frac{1-x^2}{1+y}$ ,  $\frac{1-y^2}{x+x^2}$ , and  $1 + \frac{x}{1-x}$ .

3. Extract the square root of  $8ab^3 + a^4 - 4a^2b + 4b^4$ .

4. From  $2\sqrt[3]{320}$  take  $3\sqrt[3]{40}$ .

5. Divide  $a^{-\frac{1}{2}}b^{\frac{1}{2}}$  by  $a^{\frac{1}{2}}b^{-\frac{1}{2}}$ . 6. Solve  $x^4 + 4x^2 = 12$ .

7. Solve  $x^2 - x\sqrt{3} = x - \frac{1}{2}\sqrt{3}$ .

8. What two numbers are those whose sum is  $2a$ , and the sum of their squares is  $2b$ ?

9. What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5 respectively?

10. Find three geometrical means between 2 and 162.

11. Expand to four terms  $\frac{m}{\sqrt{x^2 + a^2}}$ .

546. 1. Divide  $12x^4 - 192$  by  $3x - 6$ .

2. Divide  $x^3 + \frac{x^2}{a-b}$  by  $\frac{ab}{a-b} - m$ .

3. Solve the equation  $21 + \frac{3x-11}{16} = \frac{5x-5}{8} + \frac{27-7x}{2}$ .

4. Find the product of  $a^{\frac{1}{2}}$ ,  $a^{\frac{3}{2}}$ ,  $a^{\frac{5}{2}}$ , and  $a^{-\frac{1}{2}}$ .

5. Solve the equation  $\frac{15}{x} - \frac{72-6x}{2x^2} = 2$ .

6. Find 6 arithmetical means between 1 and 50.

7. How many different combinations may be formed of eight letters taken four at a time?

8. Expand  $(a - b)^{-\frac{1}{2}}$  to four terms.

9. Divide 150 into two such parts that the smaller may be to the greater, as 7 to 8.

10. Given  $5x + 2y = 29$  and  $2y - x = -1$ , to find  $x$  and  $y$ .

547. 1. Solve the equation  $\frac{2}{2+y} - \frac{1}{y-2} + \frac{1}{3} = 0$ .

2. What is the relation between  $a$ ,  $a^\circ$ , and  $a^{-1}$ ?

3. Find two numbers such that the sum of  $\frac{1}{2}$  the first and  $\frac{1}{3}$  of the second equals 11, and also equals three times the first diminished by the second.

4. Give the first three and last three terms of  $\left(2a - \frac{b}{3}\right)^8$ .

5. Find the *g. c. d.* of  $a^3 - b^3$  and  $a^3 - 2ab + b^3$ .

6. Find the *l. c. m.* of  $(a^2 - x^2)$ , and  $4(a - x)$ , and  $(a + x)$ .

7. Add  $\frac{13a - 29b}{5(a - b)}$ ,  $-\frac{7b - 21a}{5(a - b)}$ , and  $\frac{9b - 11a}{5(a - b)}$ .

8. Find the value of  $x$  in  $\sqrt{x + a} = \sqrt{x} + a$ .

548. 1. Reduce  $\frac{a^3 + a^2x - ax^2 - x^3}{a^2 - x^2}$  to its lowest terms.

2. Multiply  $a^{-\frac{1}{2}}b^{\frac{1}{3}}$  by  $\frac{\sqrt{b}}{\sqrt[3]{a^2}}$ ; and divide  $a^{-\frac{1}{2}}b^{\frac{1}{3}}$  by  $\frac{\sqrt{b}}{\sqrt[3]{a^2}}$ .

3. Solve the equation  $\frac{7x - 6}{35} - \frac{x - 5}{6x - 101} = \frac{x}{5}$ .

4. Given  $\frac{7x + 9}{4} - \left(x - \frac{2x - 1}{9}\right) = 7$ .

5. Solve the equation  $\frac{x^2}{2} - \frac{x}{3} + 7\frac{1}{2} = 8$ .

6. It is required to find three numbers such that the product of the first and second may be 15, the product of the first and third 21, and the sum of the squares of the second and third 74.

7. Find the sum of  $n$  terms of the series 1, 2, 3, 4, 5, 6, etc.

8. Expand to five terms  $(a^3 - b^3)^{-\frac{1}{2}}$ .

9. Find the sum of the radicals  $\sqrt{300}$  and  $\sqrt{75}$ .

10. Solve the quadratic  $\frac{x}{7} + \frac{21}{x+5} = 2\frac{1}{2}$ .

**549.** 1. Find the sum and difference of  $\sqrt{18a^5b^3}$  and  $\sqrt{50a^3b^5}$ .

2. Multiply  $2\sqrt{3} - \sqrt{-5}$  by  $4\sqrt{3} - 2\sqrt{-5}$ .

3. Solve the equation  $\frac{x-1}{7} + \frac{23-x}{5} = 7 - \frac{4+x}{4}$ .

4. Solve the equation  $\frac{x-3}{x-2} - \frac{x-4}{x-1} = \frac{7}{20}$ .

5. The sum of an arithmetical progression is 198; its first term is 2 and last term 42; find the common difference and the number of terms.

6. Expand to four terms  $(a^3 - b^3)^{\frac{1}{2}}$ .

7. Simplify the radical  $(a^3 - 2a^2b + ab^2)^{\frac{1}{2}}$ .

8. A and B together can do a piece of work in  $3\frac{1}{2}$  days, B and C in  $4\frac{1}{2}$  days, and C and A in 6 days. Required the time in which either can do it alone, and all together.

9. Find 3 numbers such that the prod. of the first and second may be 15, the prod. of the first and third 21, and the sum of the squares of the second and third 74.

**550.** 1. Given  $\frac{1}{x} + \frac{1}{y} = 2$ ,  $\frac{1}{x} + \frac{1}{z} = 3$ ,  $\frac{1}{y} + \frac{1}{z} = 3$ ; find  $x$ ,  $y$ , and  $z$ .

2. Given  $\frac{3}{8-x} - \frac{8-x}{3} = \frac{x-11}{12}$ , to find  $x$ .

3. Find the *l. c. m.* and *g. c. d.* of  $x^2 + 4x - 21$  and  $x^2 - x - 56$ .

4. It takes A 10 days longer to do a piece of work than it takes B, and both together can do it in 12 days. In how many days can each do it alone?

5. Substitute  $y + 3$  for  $x$  in  $x^4 - x^3 + 2x^2 - 3$ ; simplify and arrange the result.

# ANSWERS.

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## INTRODUCTION.

### *Page 15.*

- 1, 2. Given.
3. 2 cts. A, 6 cts. O.
4. \$8, h.; \$32, c.
5. 9 and 27.
6. 4p, C; 8p, B; 16p, A.
7. 12y, son; 36y, father.

### *Page 16.*

8. \$20, B's; \$80, A's.
9. 15, 30, 45.
10. \$7, calf; \$56, cow.
11. \$5.25, bridle;  
\$10.50, saddle;  
\$110.25, horse.
12. \$3000, daughter;  
\$6000, son;  
\$27000, wife.
13. 234, 702, 936.

### *Page 19.*

- 1-3. Given.
4.  $98\frac{1}{2}$ .
5. 18.

6. 10.
7. 34
8. 17.

### *Page 21.*

1. 60.
2. 40.
3.  $ac + 8b$ .
4.  $5b - 2d$ .
5. 35.
6. 24.
7.  $3x + 2y + ab$ .
8.  $6b - 7cx + 3a$ .
9.  $bxy + cxy$ .

### *Page 22.*

10.  $\frac{15xy}{2z} + a$ .
11.  $\frac{b-a}{xy} + 2z$ .
12.  $3x + xy + 6yz$ .
13.  $\frac{ax - ay - bx + by}{d}$ .
14. 92.
15. 120.

## ADDITION.

*Page 24.*

- 1, 2. Given.
3.  $21ab$ .
4.  $17xy$ .
5.  $15a^2$ .
6.  $-23bcd$ .
7.  $-16x^2y^2$ .
8.  $45ab^2$ .
9.  $-39abx^2y^2$ .
10.  $29b^2dm^3$ .
11. Given.
12. 4.
13. 5.

*Page 25.*

- 14, 15. Given.
16.  $8x$ .
17.  $abc$ .
18.  $-12b$ .
19.  $-12y$ .
20.  $-2m$ .
21. 1.
22. 75.
23. Given.

*Page 26.*

1.  $24a + 2b - 3d$ .
2.  $16mn - xy + bc$ .
3.  $16bc + xy - mn$ .
4.  $4ab - 3mn + 2z$ .
5.  $15xy + ab + b$ .
6. Given.
7.  $21(a + b)$ .
8.  $19c(x - y)$ .
9.  $7a\sqrt{xy}$ .
10.  $6\sqrt{a}$ .
11.  $10\sqrt{x - y}$ .

*Page 27.*

12. Given.
13.  $a(7 - 6b + 3d - 3m)$ .
14.  $y(ab + 3 - 2c - 5m)$ .
15.  $m(9 + ab - 7c + 3d)$ .
16.  $x(13a - 3b + c - 3d + m)$ .
17.  $xy(a + b - c)$ .
1. Given.

2. 16 cts., b;  
30 cts., k.

*Page 28.*

3. 26 peaches;  
49 pears.
4. 15 and 70.
5. 15 b., 25 g.
6. 1.
7. 7.
8. 7.
9. 356, A; 94, B.
10. 36 and 141.
11. 8.
12. 9 cts., top;  
23 cts., ball.
13. \$9, b; \$31, s.
14. 26 cts., A. M.;  
74 cts., P. M.
15. Given.
16. 14.
17. 7.
18. 12.
19. 60.
20. 20.

## SUBTRACTION.

*Page 31.*

- 1, 2. Given.
3.  $14xyz$ .
4.  $-62ab$ .
5.  $19ab$ .

*Page 32.*

6.  $27xy$ .
7.  $43ac$ .
8.  $37ax^2$ .
9.  $51a^2b$ .
10.  $-44x^2y^2$ .
11.  $38a^2b$ .
12. 0.
13.  $-77m^2x$ .
14.  $53x^2y$ .

15. \$150.
16.  $25^\circ$ .
17. \$420.
18.  $4xy - 6a$ .
19.  $13b^2 + 16am$ .
20.  $18x^2 + y^2 + 6a$ .
21.  $13ab + d - x$   
 $- 5m + 3n$ .
22.  $9cd - ab - 2m$   
 $+ 3n + 4y$ .
23.  $18m - 23$ .
24.  $12x^2 - 13x$ .
25.  $16ab + 13c + d$ .
26.  $a - b + c$ .
27.  $6(a + b)$ .

28.  $9(a - b + x)$ .
29.  $5(a + b)$ .
30.  $-7(x^2 - y)$ .
31. \$600, A's.
32.  $60^\circ$ .

*Page 33.*

33. Given.
34.  $(2b - c + d)x^2$ .
35.  $(ab - c - d$   
 $+ x)y$ .
36.  $a^2(7 - b + c)$ .
37.  $x(ab - 3c - d$   
 $- m)$ .

38.  $xy(8 - ab + c$   
 $- d)$ .
39.  $c(2a + bm + d)$ .

*Page 34.*

- 1-3. Given.
4.  $b - c + d - m$ .
5.  $5x + y - ab$   
 $+ 4d$ .
6.  $2a - b - c + x$   
 $+ y + d$ .
7.  $a - b + c - a$   
 $+ c + c - a + b$ .

MULTIPLICATION.

*Page 36.*

- 1-4. Given.
5.  $42abc$ .
6.  $35abcxy$ .
7.  $8dmxy$ .
8.  $63bcdxyz$ .
9.  $56abxy$ .
10.  $42acd$ .
11.  $54bcdm$ .
12.  $63adxyz$ .

*Page 37.*

13. Given.
14.  $-45abxy$ .
15.  $42abcd$ .
16.  $152abcxy$ .
17.  $-414abcxy$ .
18.  $945bcdxy$ .

*Page 38.*

- 19-21. Given.
22.  $15x^3y^2$ .
23.  $24a^3b^3$ .
24.  $a^2x^5y^2$ .
25.  $a^2b^{m+n}$ .
26.  $6x^2y^2z$ .
27.  $18a^5b^4c^2$ .
28. 18.
29. 240.
30.  $6x^2y$ .
31.  $-18a^4b^3c$ .

*Page 39.*

32.  $4x^3y^2$ .
33.  $21a^3b^3$ .
34.  $40c^4x^2y$ .
35.  $-28a^4b^3$ .

36.  $-6x^2y^3$ .
37.  $21a^2b^2c^4$ .
38.  $-28a^3c^4$ .
39.  $x^2y^2z^3$ .
- 1, 2. Given.
3.  $6acx^2 + 8c^2d$ .
4.  $15a^2b^2x - 6acd$   
 $+ 3ax^2$ .
5.  $-8a^2bd$   
 $+ 6ab^2d - 2bdm^3$ .
6.  $-15a^4c$   
 $+ 20a^2b^2c + 10a^2c^3$ .
- 7, 8. Given.

*Page 40.*

1.  $6ax + 3bx$   
 $+ 2ay + by$ .

*Page 40.*

2.  $3ax + 4ay - 3bx - 4by.$
3.  $12bd - 3cd - 4ab + ac.$
4.  $6bxy - 2ab + 6cxy - 2ac.$
5.  $3ax + 4bx - cx - 3ay - 4by + cy.$
6.  $5ax + 3ay + az + 5bx + 3by + bz.$
7.  $14cdmx - 6abm - 2icdnx + 9abn.$
8.  $24abcx + 12mx - 32abcy - 16my.$
11.  $3abc^2xyz^m.$
12.  $11abc^3d^{m+n}.$
13.  $a^2x^{2+n} + 3^{2+n}.$
14.  $cx(a+b)^4.$
15.  $5c(a-b)^5.$
16.  $abc(x+y)^{m+n}.$
17.  $-3x(a+b)^4.$

*Page 41.*

20.  $a^3 + b^3.$
21.  $a^4 + a^2b^2 + b^4.$
22.  $x^4 + x^2 + 1.$
23.  $3x^4 + 4x^3y - 13x^2 - 4x^2y^2 + 22xy - 30.$

24.  $24a^2x^2 - 6a^2y^2.$
25.  $a^3 + bdx + cdx + bcx^2.$
26. Given.
27.  $x^2 - y^2.$
28.  $a^4 + a^3 + a + 1.$
29.  $x^3 + 3x^2y + 3xy^2 + y^3.$
30.  $a^{2m} + 2a^mb^n + b^{2m}.$
31.  $x^3 + 2xy + 2xz + y^2 + 2yz + z^2.$

*Page 43.*

1.  $a^3 + 2a + 1.$
2.  $4a^3 + 4a + 1.$
3.  $4a^3 - 4ab + b^2.$
4.  $x^3 + 2xy + y^2.$
5.  $x^2 - 2xy + y^2.$
6.  $1 - x^2.$
7.  $49y^4 - 14y^3 + y^2.$
8.  $16m^2 - 9n^2.$
9.  $x^4 - y^2.$
10.  $1 - 49x^2.$
11.  $16x^2 - 8x + 1.$
12.  $25b^2 + 10b + 1.$
13.  $1 - 2x + x^2.$
14.  $1 + 4x + 4x^2.$
15.  $64b^2 - 48ab + 9a^2.$

16.  $a^2b^2 + 2abcd + c^2a^2.$
17.  $9a^3 - 4y^2.$
18.  $x^4 - y^2.$
19.  $x^3 - 2xy^2 + y^4.$
20.  $4a^4 - x^2.$

*Page 45.*

- 1, 2. Given.
3. 36.
4. 24 chickens.
5. Given.
6. 48.
7. 1960.
8. \$960.
9. 25.
10. 16.
11. 56.
12. 77.
13. 36 apples.
14. 70 sheep; 100, both.
15. 16 and 12.
16. 24 plums.
17. 42.
18. 144.
19.  $21\frac{3}{4}$ ;  $14\frac{1}{2}$ .
20.  $14\frac{7}{10}$  bu., one,  $6\frac{1}{10}$  bu., other

## DIVISION.

*Page 47.*

- 1, 2. Given.
3.  $2ab.$
4.  $5xy.$
5. 5.
6.  $2a.$
7.  $5ab.$
8.  $3bc.$
9.  $4mn.$

10, 11. Given.

12.  $-3c$ .13.  $7b$ .14.  $5d$ .15.  $-6b$ .16.  $7df$ .17.  $-9ag$ .*Page 48.*

18. Given.

19.  $d^4$ .20.  $x^2$ .21.  $c^5$ .22.  $z^4$ .23.  $4b$ .24.  $\frac{2x}{y}$ .

25. Given.

26.  $8abc^2$ .27.  $-6xz$ .28.  $5ab$ .29.  $7x^2y$ .30.  $abc$ .31.  $2abc$ .32.  $8x^2y^2z$ .33.  $8a^2bc$ .34.  $12d^3x^2y$ .35.  $\frac{12x^2}{a}$ .36.  $12x^2z^2$ .37.  $11m^2n$ .*Page 49.*

1-3. Given.

4.  $b^2+c^2+d^4$ .5.  $3x+5$ .6.  $3bc-1+4b$ .7.  $2by^2-\frac{y}{2}$ .8.  $-2x+y$ .9.  $y^2+z-1$ .10.  $-5a-4b+6$ .11.  $3ab-3a$ .12.  $-4x^2-3a^2+ax$ .13.  $a^3-5a+2b$ .14.  $1+5a-9ad$ .15.  $2a-4b-5c$ .16.  $2(a+b)^2+3x(a+b)^2$ .17.  $9x-9y$ .18.  $x(b-c)-a(b-c)$ .19.  $3a^2-2a$ .20.  $a-a^2+a^3$ .*Page 51.*

1, 2. Given.

3.  $x+y$ .4.  $a-b$ .5.  $a^2-2ab+b^2$ .6.  $c+d$ .7.  $x-d$ .8.  $2x+3y$ .9.  $a-b$ .10.  $x+y$ .11.  $a^2+ab+b^2$ .12.  $3a+2b$ .13.  $a+2$ .14.  $a^2-2ax+x^2$ .15.  $2x^3+4x^2+8x+16$ .16.  $x+5$ .17.  $x-2$ .18.  $c-x$ .19.  $a+b$ .20.  $2(a-b)$ .1. 10 yrs., son;  
46 yrs., father.2. 15, F.'s m.;  
45, J.'s m.

3. 12 and 60.

4. 12 and 45 p.

5. 31 cts., 1st;  
62 cts., 2d;  
97 cts., 3d.*Page 52.*6. 20 cows;  
180 sheep.7.  $13\frac{1}{2}$ , less;  
 $43\frac{1}{2}$ , greater.

8. 9.

9. 5 hours.

10. 8.

11. 7.

12. 5 of each.

13. 4 hours.

14.  $33\frac{3}{4}$ .

15. 10.

16. 7 m., A's No.;  
14 m., B's;  
21 m., C's.

17. \$2x, A's m.;	18. 5, 15, and 20.	20. 8, 16, 24.
\$4x, B's m.;	19. 10, A's;	21. 24.
\$8x, C's m.;	20, B's;	
\$14x, all.	30, C's	

## FACTORING.

*Page 54.*

- 1, 2. Given.
3. 2, 3, 3,  $aabb$ .
4. 2, 2,  $5bxxxyy$ .
5. 5, 7,  $aaabbcc$ .
6. 3, 7,  $xyyzzz$ .
7.  $17xyyyz$ .
8. 5, 5,  $abbcxxx$ .
9. 7,  $11aabccd$ .
10. 5, 13,  $mmnnnx$ .

*Page 55.*

- 1-3. Given.
4.  $b(y + c + 3x)$ .
5.  $2a(x + y - 2z)$ .
6.  $3bc(x - 2x - a)$ .
7.  $8dm(n - 3)$ .
8.  $7a(5m + 2x)$ .
9.  $27d(bx - 2my)$ .
10.  $3a^2(2b + 3c)$ .
11.  $7axy(3x + 5)$ .
12.  $5(5 + 3x^2 - 4x^2y^2)$ .
13.  $x(1 + x + x^2)$ .
14.  $3(x + 2 - 3y)$ .
15.  $19a^5(x - 1)$ .

*Page 56.*

- 1, 2. Given.
3.  $(a + b)(a + b)$ .

4.  $(x - y)(x - y)$ .
5.  $(m + 2n)(m + 2n)$ .
6.  $(4a + 1)(4a + 1)$ .
7.  $(7 + 5)(7 + 5)$ .
8.  $(2a - 3b)(2a - 3b)$ .
9.  $(y + 1)(y + 1)$ .
10.  $(1 - c^2)(1 - c^2)$ .
11.  $(x^m + y^n)(x^m + y^n)$ .
12.  $(2a^n - 1)(2a^n - 1)$ .
13.  $(a^2 + b^2)(a^2 + b^2)$ .
14.  $(ax^2 + y)(ax^2 + y)$ .

*Page 57.*

1. Given.
2.  $(a + x)(a - x)$ .
3.  $(3x + 4y)(3x - 4y)$ .
4.  $(y + 2)(y - 2)$ .
5.  $(3 + x)(3 - x)$ .
6.  $(a + 1)(a - 1)$ .
7.  $(1 + b)(1 - b)$ .
8.  $(5a + 4b)(5a - 4b)$ .
9.  $(2x + y)(2x - y)$ .
10.  $(1 + 4a)(1 - 4a)$ .
11.  $(5 + 1)(5 - 1)$ .
12.  $(x^2 + y^2)(x^2 - y^2)$ .
13.  $(ax + by)(ax - by)$ .
14.  $(m^2 + n^2)(m^2 - n^2)$ .
15.  $(a^m + b^n)(a^m - b^n)$ .

*Page 59.*

1. Given.
2.  $(x-1)(x^2+x+1)$ .
3.  $(x-y)(x^3+x^2y+x^2y^2+x^2y^3+xy^4+y^5)$ .
4.  $(x-1)(x+1)$ .
5.  $(1-6y)(1+6y)$ .
6. Given.
7.  $(b-x)(b+x)$ .
8.  $(d+z)(d^3-d^2z+dz^2-z^3)$ .
9.  $(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)$ .
10.  $(x+1)(x^3-x^2+x-1)$ .
11.  $(1+a)(1-a+a^2-a^3+a^4-a^5)$ .

$$12. (a+1)(a^7-a^6+a^5-a^4+a^3-a^2+a-1).$$

13. Given.

*Page 60.*

14.  $(x+y)(x^4-x^2y+x^2y^2-xy^3+y^4)$ .
15.  $(a+1)(a^3-a+a+1)$ .
16.  $(a+1)(a^4-a^3+a^2-a+1)$ .
17.  $(1+y)(1-y+y^2)$ .
18.  $(1+a)(1-a+a^2-a^3+a^4)$ .
19.  $(1+b)(1-b+b^2-b^3+b^4-b^5+b^6)$ .
- 20-28. Given.

DIVISORS.

*Page 61.*

- 1, 2. Given.
3.  $x$ .
4.  $b$ .
5.  $ac$ .
6.  $2x$ .
7.  $7m$ .
8.  $6ab$ .

*Page 63.*

- 1, 2. Given.
3.  $3ac$ .
4.  $2axy$ .
5.  $4a^3x^2z^2$ .
6.  $6ax^2z^2$ .

*Page 66.*

- 1-5. Given.
6.  $x-y$ .

7.  $a+b$ .
8.  $b+2$ .
9.  $x+3$ .
10.  $a-2$ .
11.  $a+3$ .
12.  $x+1$ .
13.  $a-b$ .
14.  $a-b$ .
15.  $x^3+3x^2+3x+1$ .

MULTIPLES.

*Page 68.*

- 1, 2. Given.
3.  $56a^4b^2c^3d$ .
4.  $80x^2y^4z^5$ .
5.  $90a^3b^4c^5$ .
6.  $420a^4b^4$ .

7.  $315x^4y^3z^5$ .
8.  $84m^3n^2y^4$ .

*Page 69.*

- 9-11. Given.
12.  $a^4+a^3b-ab^3-b^4$ .

13.  $x^3+x^2-x-1$ .
14.  $6a^3+11a^2-3a-2$ .
15.  $m^4+2m^3-m-2$ .

## REDUCTION OF FRACTIONS.

*Page 74.*

1-3. Given.

4.  $\frac{1}{3xy}$ .

5.  $\frac{3ac}{d}$ .

6.  $\frac{1}{3}$ .

7.  $\frac{7abc^2}{9x^4y^3}$ .

8.  $\frac{a-b}{a+b}$ .

9.  $\frac{x+y}{x-y}$ .

10.  $\frac{3y-3x}{2x-2z}$ .

11.  $\frac{1}{x}$ .

12.  $\frac{1}{x^2+y^2}$ .

13.  $\frac{x}{a+x}$ .

14.  $\frac{1}{a-1}$ .

15.  $\frac{1}{x+y}$ .

*Page 75.*

1. Given.

2.  $a-x$ .

3.  $b-\frac{b^2}{a}$ .

4.  $b-c$ .

5.  $b+c+\frac{2c^2}{b-c}$ .

6.  $a-b$ .

7.  $b+\frac{2ab}{b-a}$ .

8.  $a+x+\frac{a}{a-x}$ .

9.  $3x+1-\frac{3y}{4x}$ .

1. 2. Given.

3.  $\frac{4xy-b}{y}$ .

4.  $\frac{10bd+a-c}{2b}$ .

5.  $\frac{a^2+2ab+b^2+2x}{a+b}$ .

6.  $\frac{x^2-x}{x+1}$ .

7.  $\frac{12ac-a+b}{3c}$ .

8.  $\frac{39x^2+3a}{5x}$ .

*Page 76.*

1. Given.

2.  $\frac{12mx}{6m}$ .

3.  $\frac{24a^2bx}{4ab}$ .

4.  $\frac{18ac^3+24bc^2}{6c^2}$ .

5.  $\frac{x^2-y^2}{x+y}$ .

6.  $\frac{6a^2x^2y-4bx^2y}{3a^2-2b}$ .

*Page 77.*

1, 2. Given.

3.  $\frac{ab}{ac}$ .

4.  $\frac{21a^2}{49a}$ .

5.  $\frac{x^2-y^2}{x^2-2xy+y^2}$ .

6.  $\frac{32a^3(x+y)}{8a^2(x+y)^2}$ .

## Page 78.

1, 2. Given.

$$3. \frac{2cx}{2dx}, \frac{2bd}{2dx}, \frac{d^2x}{2dx}.$$

$$4. \frac{ac^2y}{2c^2xy}, \frac{2bxy}{2c^2xy}, \frac{2c^2x^2}{2c^2xy}.$$

$$5. \frac{2a^2 + 2ab}{3ab + 3b^2}, \frac{3bx}{3ab + 3b^2}.$$

$$6. \frac{x^2 - 2xy + y^2}{x^2 - y^2},$$

$$\frac{x^2 + 2xy + y^2}{x^2 - y^2}.$$

$$7. \frac{a^2 + ab}{3a}, \frac{15a - 3}{3a}.$$

$$8. \frac{bdx}{bd}, \frac{2ad}{bd}, \frac{bc + b}{bd}.$$

$$9. \frac{2b^2c - 2b^2d}{3b^2c - 3b^2d}, \frac{3ac - 3ad}{3b^2c - 3b^2d},$$

$$\frac{3b^2c + 3b^2d}{3b^2c - 3b^2d}.$$

$$10. \frac{2bxy}{2bz}, \frac{bz}{2bz}, \frac{4az}{2bz}.$$

$$11. \frac{ax + ay}{2x + 2y}, \frac{6x + 6y}{2x + 2y},$$

$$\frac{2x^2 + 2y^2}{2x + 2y}.$$

$$12. \frac{a^2 - 2ax + x^2}{a^2 - x^2},$$

$$\frac{a^2 + 2ax + x^2}{a^2 - x^2}.$$

## Page 79.

1. Given.

$$2. \frac{2acx}{4bcx}, \frac{4b^2c^2}{4bcx}, \frac{bxy}{4bcx}.$$

$$3. \frac{3c^2d}{3abc}, \frac{2bcx}{3abc}, \frac{3bxy}{3abc}.$$

$$4. \frac{6ay}{12y}, \frac{4by}{12y}, \frac{3cy}{12y}, \frac{12x}{12y}.$$

$$5. \frac{4abc^2}{4b^2c}, \frac{8cd}{4b^2c}, \frac{bx^2y}{4b^2c}.$$

$$6. \frac{16a^2b}{24a^2c}, \frac{18a^2c}{24a^2c}, \frac{24x}{24a^2c},$$

$$\frac{3a^2c}{24a^2c}.$$

$$7. \frac{ac}{2bc}, \frac{2cd}{2bc}, \frac{2xy}{2bc}.$$

$$8. \frac{(a+b)^2}{a^2 - b^2}, \frac{(a-b)^2}{a^2 - b^2}, \frac{a^2 + b^2}{a^2 - b^2}.$$

$$9. \frac{4xy(x+y)}{6xy(x+y)}, \frac{6a(x+y)}{6xy(x+y)},$$

$$\frac{abxy}{6xy(x+y)}.$$

$$10. \frac{ad}{a^2b^2}, \frac{bx}{a^2b^2}.$$

$$11. \frac{b^2cdx}{ab^2c^2d}, \frac{acdm}{ab^2c^2d}, \frac{ab^2y}{ab^2c^2d}.$$

$$12. \frac{x^2z}{xy^2z}, \frac{ayz + byz}{xy^2z}, \frac{dy^2}{xy^2z}.$$

$$13. \frac{4cmx^2 + 4cnx^2}{12a^2cx^2},$$

$$\frac{6acm - 6acn}{12a^2cx^2}, \frac{3a^2m^2x}{12a^2cx^2}.$$

## ADDITION OF FRACTIONS.

*Page 80.*

1, 2. Given.

3.  $\frac{27ac}{2xy}$ .

4.  $\frac{39dxz}{5abc}$ .

5.  $\frac{5b}{x}$ .

6.  $\frac{7a + 2b}{c}$ .

7. Given.

*Page 81.*

8. Given.

9.  $\frac{45a + 12x + 20y}{60}$ .

10.  $\frac{8ax + 6 + 9ay}{12a}$ .

11.  $\frac{ab - ac + bx + cx}{b^2 - c^2}$ .

12.  $\frac{3x - y}{2xy}$ .

13.  $\frac{2a + 2ax + 3}{ay}$ .

14.  $\frac{ax - ay + abx + aby}{x^2 - y^2}$ .

15.  $\frac{5ca^2 + 3oxy + 3bdx^2}{15dx}$ .

16.  $\frac{3ah - 2dn - d^2}{3dh}$ .

17.  $\frac{am - dy}{my}$ .

18.  $\frac{nx - mx - hy}{my - ny}$ .

19. - 6.

20.  $\frac{4adx + 6bcx - bdm}{bdx}$ .

1. Given.

2.  $a + c + \frac{bx + 2d}{2x}$ .

3.  $x + \frac{am - ay - bx + bd}{bm - by}$ .

4.  $3d + a + b - c - \frac{xy + z}{2}$ .

5.  $5x + \frac{2a - by}{2b}$ .

*Page 82.*

6. Given.

7.  $\frac{3bd + 2a}{b}$ .

8.  $\frac{a + b - 4cy}{c}$ .

9.  $\frac{x + y - a^2}{a}$ .

10.  $\frac{3x^2 - 2xy - y^2 + a - b}{x - y}$ .

11.  $\frac{x - y - a^2 + 6ab - 5b^2}{a - b}$ .

12.  $\frac{2x^2 + 2xy - 2x - 2y + a + b}{x - 1}$ .

## SUBTRACTION OF FRACTIONS.

*Page 83.*

1, 2. Given.

3.  $\frac{9abc}{d}$ .

4.  $\frac{17xyz - 5cd}{a}$ .

5, 6. Given.

7.  $\frac{ay - dm + bm}{my}$ .

8.  $\frac{by - dy + bm}{my}$ .

9.  $\frac{17d - 9a}{12}$ .

*Page 84.*

10.  $\frac{hy + hm + dm}{my}$ .

11.  $\frac{h - my}{y}$ .

12.  $a + \frac{bd + ch}{cd}$ .

13.  $\frac{6a + 5b - 2d}{6}$ .

14.  $\frac{ad + ay - bc + cx}{bd - dx + by - xy}$ .

15.  $a - \frac{2x + 3dy}{2y}$ .

16.  $\frac{x^2 - y^2 - 10a + 10b}{10x + 10y}$ .

17.  $\frac{4x - y + 3c + 6a}{6}$ .

## MULTIPLICATION OF FRACTIONS.

*Page 85.*

1-4. Given.

5.  $h + 3d$ .

6.  $\frac{ab}{4}$ .

7.  $\frac{6cx - 9cy + 4dx - 6dy}{15c + 4d}$ .

8.  $2abc$ .

9.  $\frac{a + b}{4 + 5y}$ .

10.  $\frac{2a^2 + 2a^2b}{b + 1}$ .

11.  $8x^2 + 12x$ .

12.  $2ax - 2bx + 3a - 3b$ .

13.  $abc$ .

14.  $\frac{a + b}{5}$ .

15.  $\frac{2x + y}{5}$ .

16.  $\frac{9c - 3d}{4}$ .

17.  $3xy + 3x$ .

18.  $\frac{m^2}{x - z}$ .

*Page 87.*

1, 2. Given.

3.  $6xy$ .

4.  $\frac{dx}{ay}$ .
  5.  $\frac{x^2 - y^2}{y^2z + yz^2}$ .
  6.  $\frac{3}{8a + 24x}$ .
  7.  $\frac{4hy}{3cx}$ .
  8.  $\frac{d(a+b)}{cx}$ .
  9. 10. Given.
  11.  $\frac{y - 3x}{2xy}$ .
  12.  $\frac{b - 2a}{3ab}$ .
  13.  $\frac{a^2x + a^2 + ab}{by}$ .
  14.  $\frac{xy + 2x + y^2 + 2y}{xy}$ .
  15.  $\frac{x^4 - y^4}{x^2y}$ .
  16.  $2b + 4$ .
- Page 88.*
1. Given.
  2.  $\frac{abd^2x}{y}$ .
  3.  $\frac{abd + acd}{xy}$ .
  4.  $\frac{mx + nx}{4}$ .
  5.  $\frac{4ac + 4ch}{d}$ .
  6.  $\frac{15ax - 5xy}{y}$ .
  7.  $\frac{5bx^2 + 5b}{x - 1}$ .
  8.  $\frac{7x - 7ax}{acx - acy}$ .
  9.  $\frac{3}{3}$ .
  10.  $\frac{3ac}{2}$ .
  11.  $\frac{2ax^3 + 2ax}{3x - 3}$ .
  12.  $\frac{8x^2y}{a + b}$ .
  13.  $\frac{6am}{x + 1}$ .
  14.  $\frac{2abxy + b^2xy}{4a + b}$ .
  15.  $1 - n$ .
  1.  $\frac{9cx - 3dx}{4}$ .
  2.  $3x(y + 1)$ .
  3.  $\frac{xy + 2x + y^2 + 2y}{xy}$ .
  4.  $9x$ .
  5.  $\frac{a}{6}$ .
  6.  $x$ .
  7.  $\frac{m^2}{x - z}$ .
  8.  $6a^2y^2$ .
  9.  $x^2 - y^2$ .
  10.  $\frac{2x + y}{5}$ .
  11.  $\frac{x^4 - y^4}{x^2y}$ .
  12.  $2b + 4$ .
  13.  $2a(c + d)$ .
  14.  $\frac{y - 3x}{2xy}$ .
  15.  $b^2$ .
  16.  $\frac{2(x + y)}{x}$ .

## DIVISION OF FRACTIONS.

*Page 90.*

1-4. Given.

5.  $\frac{2x}{n}$ .

6.  $\frac{2a}{3bc}$ .

7.  $1 + \frac{b}{c}$ .

8.  $\frac{a+y}{x}$ .

9.  $\frac{a}{2b}$ .

10.  $\frac{a^2 + ac + c^2}{b + c}$ .

11.  $\frac{x+y}{a+c}$ .

12.  $\frac{x+2y}{a^2 - b^2}$ .

*Page 92.*

1. Given.

2. 3 times.

3. 5.

4. Given.

5.  $\frac{a^2by}{cdx}$ .

6.  $\frac{ax^2}{4y^2}$ .

7.  $\frac{3x}{2y}$ .

8.  $\frac{2}{x-1}$ .

9.  $\frac{a^2 - a}{2}$ .

10.  $\frac{2x^2y}{3}$ .

11. 6.

12.  $\frac{a^2b}{2a + 2b}$ .

13.  $\frac{3x + 3y}{abcxy}$ .

14.  $\frac{x-a}{6bc^2}$ .

15.  $\frac{8y^2}{3ax + 3ay}$ .

16.  $\frac{4xy^2}{3ax + 3bz}$ .

17.  $\frac{bx^2}{a^2}$ .

18.  $\frac{4dy}{b}$ .

19.  $\frac{b}{4dy}$ .

20.  $\frac{4dy}{b}$ .

1. Given.

2.  $\frac{abdmxy}{cx}$ .

3.  $\frac{mx + nx}{4}$ .

4.  $\frac{ax + x^2}{5c}$ .

5.  $\frac{25x^2 - 5xy}{y}$ .

6.  $\frac{5ax^2 + 5a}{x + 1}$ .

7.  $3x - 3ax$ .

*Page 93.*

8, 9. Given.

10.  $\frac{a^2 + 2a + 1}{a^2 - 2a + 1}$ .

11.  $\frac{a^2 - b^2}{x^2 - y^2}$ .

12.  $\frac{x^2 - xy^2 + x^2y - y^2}{a - b}$ .

13.  $\frac{x^2 - y^2}{a^2 - b^2}$ .

*Page 94.*

1.  $\frac{5a}{4xyz}$ .

2.  $\frac{35bcd}{243xy^2}$ .

3.  $\frac{24xy}{cd}$ .

4.  $\frac{3a}{x + y}$ .

5.  $\frac{23xz}{17y(a + b)}$ .

- |   |                                       |
|---|---------------------------------------|
| 6. $\frac{8b(x-y)}{c}$ .                    | 11. $\frac{2c}{a^2 - ac + c^2}$ .     |
| 7. Given.                                   | 12. $\frac{4(a^2 - 2ax + x^2)}{3x}$ . |
| 8. $\frac{a}{a+x}$ .                        | 13. $\frac{1}{(a+b)^2}$ .             |
| 9. $\frac{4c}{c+z}$ .                       | 14. $\frac{x}{x^2 + 2x + 1}$ .        |
| 10. $\frac{4d(a^2 - x^2)}{3a(c^2 - x^2)}$ . | 15. $\frac{c}{x}$ .                   |

## SIMPLE EQUATIONS.

*Page 97.*

- 1, 2. Given.  
 3.  $a - b + c - d$ .  
 4.  $a + b - ab + c$ .

*Page 98.*

5. Given.  
 6.  $2 - a + b$ .  
 7.  $b + c - a - 3$ .  
 8.  $ad - bc + 2m - 8$ .  
 9.  $17 - 3ab - d$ .  
 10.  $4cd + d - 3bh - 1$ .  
 11.  $32 - c + d$ .  
 12. 11.

*Page 100.*

- 1, 2. Given.  
 3. 15.  
 4. 12.

5.  $24\frac{1}{2}$ .

6. Given.

7.  $\frac{5bd - 4ad}{3b}$ .

8.  $\frac{10a - 2b - c}{16}$ .

9.  $7\frac{1}{2}$ .

10.  $\frac{16c}{15}$ .

*Page 101.*

1. 8.  
 2. 50.  
 3. 30.  
 4. 9.  
 5. 7.  
 6. 9.  
 7. 72.  
 8. 30.  
 9. 5.  
 10. 28.  
 11. 12.

12. 24.

13. 14.

14.  $-1\frac{3}{5}$ .

15. Given.

*Page 102.*

16. 12.

17. 60.

18. 4.

19. 20.

20.  $ab + ac$ .

21.  $\frac{dn}{a}$ .

22.  $\frac{6c}{3a + 2b}$ .

23.  $\frac{ad - bc}{ac}$ .

24.  $\frac{2ab}{ac - 2c}$ .

25.  $\frac{5a + 13\frac{1}{2}}{24}$ .

$$26. \frac{24b + 150c - 20a}{45}.$$

$$27. \frac{3a - 6}{4}.$$

$$28. \frac{1}{2a - 1}.$$

$$29. \frac{a^2(c - a + c^2 - ac)}{c^2}.$$

$$30. -\frac{b^2}{2b} = -\frac{b}{2}.$$

$$31. \frac{a - 1}{2}.$$

$$32. 7b.$$

$$33. 84.$$

*Page 103.*

$$34. 20.$$

$$35. 24.$$

$$36. 1.$$

$$37. 1\frac{1}{2}.$$

$$38. 16\frac{1}{2}.$$

$$39. 1\frac{1}{23}.$$

$$40. 36.$$

$$41. 11.$$

$$42. 1200.$$

$$43. 14\frac{1}{2}.$$

$$44. 5.$$

$$45. 4\frac{1}{2}.$$

$$46. \frac{b}{2}(1 - a^2).$$

$$47. \frac{2a - 2b + c}{2b}.$$

$$48. \frac{3a - 6}{4}.$$

$$49. \frac{6d}{5b}.$$

$$50. \frac{1 - 8a}{1 + 8a}.$$

$$51. \frac{a}{4} - 2.$$

*Page 105.*

$$1. \text{ Given.}$$

$$2. \$8, \text{ vest;}$$

$$\$32, \text{ coat.}$$

$$3. \$1500, A;$$

$$\$3000, B;$$

$$\$4500, C.$$

$$4. 40 \text{ men;}$$

$$80 \text{ boys;}$$

$$880 \text{ women.}$$

$$5. 40 \text{ miles;}$$

$$80 \text{ miles.}$$

$$6. 133\frac{1}{3} \text{ barrels.}$$

$$7. 12 \text{ p., 1st;}$$

$$24 \text{ p., 2d;}$$

$$60 \text{ p., 3d.}$$

$$8. 28\frac{1}{2} \text{ feet.}$$

$$9. \$120.$$

$$10. \$50, B's \text{ sh. ;}$$

$$\$100, A's \text{ sh. ;}$$

$$\$150, C's \text{ sh.}$$

$$11. 18 \text{ yrs., w. ;}$$

$$36 \text{ yrs., m.}$$

$$12. \$3000.$$

$$13. 16 \text{ and } 41.$$

$$14. \$6000.$$

*Page 106.*

$$15. 13.$$

$$16. 30. \text{ days.}$$

$$17. 240 \text{ m., one;}$$

$$180 \text{ m., other.}$$

$$18. 12 \text{ in., one;}$$

$$16 \text{ in., other.}$$

$$19. \$25, H. ;$$

$$\$175, C.$$

$$20. 8 \text{ h., } 24 \text{ m. A. M.}$$

$$21. 9\frac{1}{3} \text{ days.}$$

$$22. 32 \text{ of each.}$$

$$23. 30, 75, \text{ and } 45.$$

$$24. 25 \text{ cts., ch. ;}$$

$$75 \text{ cts., goose;}$$

$$\$1.50, \text{ turkey.}$$

$$25. 8 \text{ ft. } 8 \text{ in.}$$

$$26. 40 \text{ and } 60.$$

$$27. \frac{ad}{c + d}, \text{ less.}$$

$$\frac{ac}{c + d}, \text{ greater.}$$

$$28. \$1128.$$

*Page 107.*

29. Given.  
 30. 60 lbs., b;  
     120 lbs., m.  
 31. 15 yrs., B's;  
     30 " A's.  
 32.  $32\frac{1}{2}$  yrs., C's;  
      $37\frac{1}{2}$  " B's;  
      $40\frac{1}{2}$  " A's.  
 33. 1175 votes d.;  
     1325 " s.  
 34. 164 artillery;  
     472 cavalry;  
     564 infantry.  
 35. \$533 $\frac{1}{2}$ , B's;  
     \$633 $\frac{1}{2}$ , A's;  
     \$833 $\frac{1}{2}$ , C's.  
 36. \$56.25, one;  
     \$93.75, other.

*Page 108.*

37. \$336, pr. one.  
     \$280, " other  
 38. 14 yrs., y'ngest;  
     16 " next;  
     18 " eldest.  
 39.  $16\frac{2}{3}$  days.  
 40. 550.  
 41. 30 and 18.  
      $\frac{12a}{13}$ .  
 42.  $\frac{12a}{13}$ .  
 43. 225 acres, A;  
     315 " B.

44.  $2\frac{2}{3}$  hrs.  
 45. 5, 1st part;  
     8, 2d "  
     2, 3d "  
     24, 4th "  
 46. 9.  
 47. 47 sheep.  
 48. \$120.

*Page 109.*

49. 60 min.  
      $\frac{d}{m - n}$ .  
 50.  $\frac{d}{m - n}$ .  
 51. 300 leaps.  
 52.  $\frac{ab}{a - c}$ , one.  
      $\frac{bc}{a - c}$ , other.  
 53. 72 lbs.  
 54. 36 hours;  
     312 miles.  
 55. 20 yrs., s;  
     40 yrs., f.  
 56. 280.  
 57. \$324, 1st;  
     \$108, 2d;  
     \$144, 3d.  
 58. Given.

*Page 110.*

59. 8, 1st part;  
     12, 2d "  
     16, 3d "

60. 9 in. and 12 in  
 61. \$75.  
 62. 27 days.  
 63. \$1575, one;  
     \$2625, other.  
 64. 12 days.  
 65. \$720.  
 66. \$384, sum;  
     \$162, A's sh.;  
     \$118, B's "  
     \$104, C's "  
 67. Given.

*Page 111.*

68. 6 and 8.  
 69. 3456, one;  
     2304, other.  
 70. 3 m. an hour.  
 71. 400 in.,  
     or 33 $\frac{1}{3}$  ft.  
 72. 8 k. of one name  
     6 k. of another;  
     3 k. " "  
     2 k. " "  
 73. 7 and 8.  
 74. 240 leaps of d.  
 75. 100 days;  
     30000 m., 1st;  
     24000 m., 2d;

## TWO UNKNOWN QUANTITIES.

*Page 114.*

1. Given.
2.  $x=8$ ,  $y=4$ .
3.  $x=12$ ,  $y=6$ .
4.  $x=18$ ,  $y=2$ .
5.  $x=1$ ,  $y=3$ .
6.  $x=16$ ,  $y=35$ .
7.  $x=3$ ,  $y=2$ .
8. Given.
9.  $x=4$ ,  $y=5$ .
10.  $x=6$ ,  $y=12$ .
11.  $x=5$ ,  $y=6$ .
12.  $x=10$ ,  $y=3$ .
13.  $x=11$ ,  $y=9$ .
14.  $x=3$ ,  $y=2$ .

*Page 116.*

- 15, 16. Given.
17.  $x=3$ ,  $y=5$ .
18.  $x=4$ ,  $y=7$ .
19.  $x=7$ ,  $y=2$ .
20.  $x=16$ ,  $y=35$ .
21.  $x=3$ ,  $y=2$ .
1.  $x=4$ ,  $y=5$ .
2.  $x=8$ ,  $y=2$ .
3.  $x=5$ ,  $y=3$ .
4.  $x=3$ ,  $y=4$ .
5.  $x=3$ ,  $y=4$ .
6.  $x=12$ ,  $y=3$ .
7.  $x=3$ ,  $y=5$ .
8.  $x=4$ ,  $y=3$ .

*Page 117.*

9.  $x=34$ ,  $y=46$ .
10.  $x=4$ ,  $y=2$ .
11.  $x=16$ ,  $y=7$ .
12.  $x=8$ ,  $y=1$ .
13.  $x=60$ ,  $y=36$ .
14.  $x=10$ ,  $y=20$ .
15.  $x=5$ ,  $y=2$ .
16.  $x=2$ ,  $y=4$ .
17.  $x=8$ ,  $y=6$ .
18.  $x=4$ ,  $y=9$ .
19.  $x=6$ ,  
 $y=12$ .
20.  $x=18$ ,  $y=14$ .
1.  $x=43$ ,  $y=27$ .
2. 4 cts., lemons;  
6 cts., oranges.
3. 233 v.; 142 v.
4. 21 and 54.
5. \$48, cow;  
\$96, horse.
6. 40 l.; 50 g.

*Page 118.*

7. 3 and 2.
8. 11111 m., one;  
9999 m., other.
9. 56.
10. \$320, B's;  
\$250, A's.

11.  $\frac{4}{15}$ .

12. \$900, A's;  
\$2400, B's.
13. 31 and 17.
14. \$6000 h.;  
\$2500 g.
15. 30 and 20.
16. \$560, B's;  
\$720, A's.
17. 25 y. and 35 y.
18. \$180, 1st;  
\$115, 2d.

*Page 119.*

19. 140 m., ship;  
160 m., steamer
20. 12 and 18.
21. 108 ft.
22. 30 yrs.;  
13 verses.
23. 10 l.; 30 g.
24. 3 oxen;  
21 colts.
25. 53.
26. \$5000, B's cap.;  
\$4800, A's "
27. \$21 or 63 g.
28.  $x = \frac{an}{n+1}$ ,  
 $y = \frac{a}{n+1}$ .

## THREE OR MORE UNKNOWN QUANTITIES.

*Page 121.*

1. Given.

2.  $x=7, y=5, z=4$ .

3.  $x=2, y=3, z=5$ .

4.  $x=8, y=4, z=2$ .

5.  $x=4, y=3, z=5$ .

6.  $x=24, y=6, z=23$ .

7.  $x=7, y=10, z=9$ .

8.  $x=24, y=60, z=120$ .

9-11. Given.

12.  $w=2, x=3, y=4, z=5$

13.  $x=2, y=3, z=4$ .

*Page 123.*

1. 12 yrs., 1st;

15 " 2d;

17 " 3d.

2. \$5, s.; \$4, l.;  
\$20, c.

3. 5, 8, and 11.

4. 630 men, 1st;

675 " 2d;

600 " 3d.

5. 50 cts., 1st;

60 " 2d;

80 " 3d.

6. 105 min., A;

210 " B;

420 " C.

7. 18 = 1st;

22 = 2d;

10 = 3d;

40 = 4th.

8. 46 m., A's;

9 " B's;

7 " C's.

9. \$64, A's;

\$72, B's;

\$84, C's.

## GENERALIZATION.

*Page 125.*

1. 3 chickens.

2. 30 rods.

3. 12.

4.  $5\frac{1}{10}$  yrs.

5. 7 ft.

6. 34.

7. \$205, \$187.

12.  $22\frac{3}{4}$  hrs.

13. \$67.32.

*Page 129.*

14. 9077.

15. \$1036.12 $\frac{1}{2}$ .

16. 587.19 bu.

17.  $12\frac{3}{4}$  per cent.

18. 40 per cent.

19. 60 per cent.

20. \$3000.

*Pages 130-133.*

21. \$37500.

22. \$31250.

23. \$2700, B's;

\$2300, C's.

24. \$5625.

25.  $1842\frac{1}{2}$  A.

26. \$55.80.

27. \$190.32.

28. \$1253.

29. \$418.60.

30. \$5250.

31. \$3865.86.

32. \$1339.29.

33. \$222.22 +.

34.  $2\frac{1}{2}$  yrs.

35. Given.

36. 3h.  $16\frac{4}{11}$  m. P.M.

37. 6h.  $32\frac{8}{11}$  m. P.M.

38. 9h.  $49\frac{1}{11}$  m. P.M.

*Pages 127, 128.*

8. \$961, A's;

\$614, B's.

9. 1248, g.;

902, l.

10.  $4\frac{3}{4}$  days.

11.  $5\frac{1}{2}$  hrs.

## INVOLUTION.

## Page 137.

2.  $a^2b^2c^2$ .
3.  $a^2b^2c^3$ .
4.  $x^3y^3z^3$ .
5.  $a^5b^5c^5$ .
6.  $16x^8y^4$ .
7.  $216a^9b^6$ .
8.  $625a^{12}b^8c^4$ .
9.  $64a^{12}b^6c^{12}$ .
10.  $a^8b^8c^8d^8$ .
11.  $x^ny^nz^n$ .

## Page 138

12.  $(a+b)^{10}$ .
13.  $(a+b)^{2n}$ .
14.  $(x-y)^{mn}$ .
15.  $(x+y)^{2n}$ .
16.  $(a^3+b^3)^2$ .
17.  $a^9b^6h^{12}$ .
19.  $\frac{27a^3b^6}{8a^3}$ .
20.  $\frac{16a^8b^4c^{12}}{x^8y^{12}}$ .

21.  $\frac{49a^{2n}b^4}{9a^4b^{2n}}$ .
22.  $\frac{2^m}{a^m}$ .
23.  $\frac{a^{mn}b^{n \times n}}{x^n y^{n \times n}}$ .
26.  $x^3 + 6x^2y + 6x^4$   
 $+ 12xy^3 + 24xy$   
 $+ 12x + 8y^3$   
 $+ 24y^3 + 24y$   
 $+ 8$ .

## Page 142.

1.  $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ .
2.  $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ .
3.  $c^7 + 7c^6d + 21c^5d^2 + 35c^4d^3 + 35c^3d^4 + 21c^2d^5 + 7cd^6 + d^7$ .
4.  $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$ .
5.  $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$ .
6.  $y^{10} + 10y^9z + 45y^8z^2 + 120y^7z^3 + 210y^6z^4 + 252y^5z^5$   
 $+ 210y^4z^6 + 120y^3z^7 + 45y^2z^8 + 10yz^9 + z^{10}$ .
7.  $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6$   
 $- 36a^2b^7 + 9ab^8 - b^9$ .
8.  $m^{11} + 11m^{10}n + 55m^9n^2 + 165m^8n^3 + 330m^7n^4 + 462m^6n^5$   
 $+ 462m^5n^6 + 330m^4n^7 + 165m^3n^8 + 55m^2n^9 + 11mn^{10} + n^{11}$ .
9.  $x^{12} - 12x^{11}y + 66x^{10}y^2 - 220x^9y^3 + 495x^8y^4 - 792x^7y^5$   
 $+ 924x^6y^6 - 792x^5y^7 + 495x^4y^8 - 220x^3y^9 + 66x^2y^{10}$   
 $- 12xy^{11} + y^{12}$ .
10.  $a^n + na^{n-1}b + n \frac{n-1}{2} a^{n-2}b^2 + n \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^3$   
 $+ n \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 + \text{etc.}$

13.  $x^3 + 3x^2 + 3x + 1.$

14.  $b^4 - 4b^3 + 6b^2 - 4b + 1.$

15.  $1 - 5a + 10a^2 - 10a^3 + 5a^4 - a^5.$

16.  $1 + na + n \frac{n-1}{2} a^2 + n \frac{n-1}{2} \times \frac{n-2}{3} a^3 + \text{etc.}$

*Pages 143, 144.*

17.  $x^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3y^2z + 3yz^2 + z^3.$

18, 19. Given.

20.  $x^2 + 2x(y + z) + y^2 + 2yz + z^2.$

21.  $a^2 - 2a(b - c) + b^2 - 2bc + c^2.$

22.  $a^2 + 2a(x + y + z) + x^2 + 2x(y + z) + y^2 + 2yz + z^2.$

23. Given.

24. 
$$\frac{9a^3 + 12a + 4}{9}.$$

25. 
$$\frac{4a^2 - 4ac + c^2}{4}.$$

26. 
$$\frac{36 - 168abc + 196a^2b^2c^2}{49}.$$

27. 
$$\frac{b^2 - 6bmxy + 9m^2x^2y^2}{m^2}.$$

28, 29. Given.

## MULTIPLICATION AND DIVISION OF POWERS.

*Pages 145, 146.*

1, 2. Given.

3.  $a^8.$

4.  $x^{-6}.$

5.  $b^{-3}.$

6.  $a^{m+n}.$

7.  $a^{-7}b^5.$

8.  $a^3c^4a^8.$

9.  $b^2c^2y^2.$

10.  $a^3y^{-9}z^7.$

11. Given.

12.  $a^{11}.$

13.  $x^{-11}.$

14.  $b^8.$

15.  $c^{-2}.$

16.  $x^7y^{-8}z^5.$

17.  $4abc^{-4}.$

18.  $3x^6y.$

19.  $12a^5c^3.$

20-23. Given.

24.  $\frac{a}{x^2y}.$

25.  $\frac{ay^4}{b}.$

26.  $\frac{a}{a^3x^2}.$

27.  $\frac{bx^{-n}}{a}.$

## EVOLUTION.

*Pages 148, 149.*

13.  $a^{\frac{1}{2}}.$

14.  $x^{\frac{5}{4}}.$

15.  $y^{\frac{1}{3}}.$

16.  $a^{25}.$

17.  $a^{-2}.$

18.  $a^{-6}.$

19.  $b^{-3}.$

20.  $x^{14}.$

21.  $y^{-10}.$

22-25. Given.

**Pages 151, 152.**

- 1, 2. Given.
3.  $a^3$ .
4.  $a^{\frac{1}{2}}$ .
5.  $4^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$ .
6.  $2a^{\frac{1}{2}}b^{\frac{1}{2}}$ .
7.  $3a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$ .
8.  $2a^m$ .
9.  $3^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{1}{2}}$ .
10.  $6a^2b$ .
11.  $2^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$ .
12.  $8a^{\frac{1}{2}}b^{\frac{1}{2}}$ .

13.  $(13)^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$ .
14.  $7x^2y^3$ .
15.  $3a^{\frac{1}{2}}b^{\frac{1}{2}}$ .
16.  $\frac{7x^2}{8y}$ .
1. Given.
2.  $x + 2$ .
3.  $a - 1$ .
4.  $1 + x$ .
5.  $x + \frac{2}{3}$ .
6.  $a - \frac{1}{2}$ .
7.  $x + \frac{b}{2}$ .

**Page 153.**

8. Given.
9.  $x + y + z$ .
10.  $a - 2b + 1$ .
11.  $a^2 + 2b - 2$ .
12.  $1 - 2b^2 + x$ .
13.  $2a^2 - 4a + 2$ .
14.  $a - \frac{b}{2}$ .
15.  $\frac{x}{y} - \frac{y}{x}$ .

## RADICAL QUANTITIES.

**Page 156.**

- 1-5. Given.
6.  $a\sqrt{b}$ .
7.  $2a\sqrt{2b}$ .
8.  $6\sqrt{xy}$ .
9.  $6\sqrt[3]{3}$ .
10.  $15\sqrt[3]{5}$ .
11.  $36a\sqrt{7b}$ .
12.  $3a\sqrt[3]{2c}$ .
13.  $21a\sqrt{1 - 3b}$ .
14.  $4x\sqrt[3]{y}$ .
15.  $3a\sqrt[4]{b}$ .
16.  $6a\sqrt{13c}$ .
17.  $12a\sqrt{11}$ .

**Page 157.**

1. Given.
2.  $\sqrt[3]{8a^3b^3}$ .

3.  $\sqrt{(2a + b)^2}$ .
4.  $\sqrt{(a - 2b)^3}$ .
5.  $\sqrt{9a^2b}$ .
6.  $\sqrt[3]{8a^3b}$ .
7.  $\sqrt[4]{16x^2y^{\frac{1}{2}}z}$ .
8.  $\sqrt[3]{\frac{a^3b^3c^3}{27}}$ .
9.  $\sqrt[3]{27(a - b)^3}$ .
10.  $\sqrt[3]{a^6}$ .
11.  $\sqrt[4]{a^{12}c^8}$ .
12.  $\sqrt{(a - b)^2}$ .
13.  $\sqrt[n]{a^{mn}}$ .

**Pages 158, 159.**

1. Given.
2.  $(a^3)^{\frac{1}{2}}, (b^4c^4)^{\frac{1}{2}}$ .
3.  $9^{\frac{1}{2}}, (125)^{\frac{1}{3}}$ .

4.  $(a^8)^{\frac{1}{2}}, (1296)^{\frac{1}{3}}$ .
5.  $\sqrt[13]{15625}, \sqrt[13]{81}, \sqrt[13]{8}$ .
6.  $\sqrt[6]{4x^8}, \sqrt[6]{125x^9}$ .
7.  $\sqrt[6]{64a^9}, \sqrt[6]{4a^4}$ .
8.  $\sqrt[m]{a^n}, \sqrt[m]{b^m}$ .
9.  $(b^n)^{\frac{1}{2m}}, (c^2)^{\frac{1}{2m}}$ .
10.  $\sqrt[6]{(a + b)^3}, \sqrt[6]{(a - b)^3}$ .
11.  $\sqrt[12]{(x - y)^8}, \sqrt[12]{(x + y)^9}$ .
- 1, 2. Given.
3.  $(3^{\frac{1}{2}})^{\frac{1}{2}}, (4^{\frac{1}{2}})^{\frac{1}{2}}$ .
4.  $(a^{10})^{\frac{1}{2}}, (b^{15})^{\frac{1}{2}}$ .
5.  $(a^4)^{\frac{1}{2}}, (b^{\frac{1}{2}})^{\frac{1}{2}}$ .
6.  $(a^{\frac{1}{2}})^{\frac{1}{2}}, (b^{\frac{1}{2}})^{\frac{1}{2}}$ .
7.  $(a^{\frac{1}{2}})^{\frac{1}{2}}, (b^{\frac{1}{2}})^{\frac{1}{2}}$ .

## ADDITION OF RADICALS.

**Page 160.**

1-3. Given.

4.  $5\sqrt{3}$ .

5.  $2\sqrt{5} + 4\sqrt{3}$ .

6.  $(2b + 3a)\sqrt{b}$ .

7.  $(a^2 + 3c)\sqrt{3ab}$ .

8.  $(9x + 8a)\sqrt{2a}$ .

9.  $25\sqrt[3]{2}$ .

10.  $118\sqrt{3}$ .

11.  $30a\sqrt{b}$ .

12.  $(5bx + 6x^2)\sqrt{a}$ .

13.  $4x^2\sqrt[3]{y} + 5x\sqrt{xy}$ .

## SUBTRACTION OF RADICALS.

**Page 161.**

1. Given.

2.  $8\sqrt{7}$ .

3.  $4\sqrt{30} - 12\sqrt{7}$ .

4.  $52\sqrt{5}$ .

5.  $(21x - 10)\sqrt{ax}$ .

6.  $2\sqrt[3]{a + b}$ .

7.  $7\sqrt[3]{b}$ .

8.  $9b\sqrt[3]{2bx}$ .

9.  $a^{-\frac{1}{2}}$ .

10.  $1\frac{5}{8}\sqrt{3}$ .

## MULTIPLICATION OF RADICALS.

**Page 162.**

1-3. Given.

4.  $90\sqrt{10}$ .

5.  $abx$ .

6.  $\sqrt{a^2 - b^2}$ .

7.  $\sqrt{acxy}$ .

8.  $a\sqrt{ac}$ .

9. Given.

10.  $\sqrt[n]{a^n x^n}$ .

11.  $42\sqrt[3]{2}$ .

12.  $12a$ .

13. 6.

14.  $4ax$ .

15. Given.

16.  $\sqrt{5}$ .

17.  $2\sqrt{5}$ .

18.  $(m + n)\sqrt[3]{m + n}$ .

19.  $a\sqrt{\frac{3d}{c}}$ .

## DIVISION OF RADICALS.

**Page 164.**

4.  $\sqrt{3a^3}$ , or

$a\sqrt{3a}$ .

5.  $3\sqrt{bx}$ .

6.  $(a^2 + x)^{\frac{1}{2}}$ .

7.  $12(ay)^{\frac{1}{2}}$ .

8.  $3b\sqrt{x}$ .

9.  $\frac{15a}{2}\sqrt{b}$ .

10.  $2a\sqrt{x}$ .

11.  $(a + b)^{\frac{1}{2}}$ .

12.  $15x\sqrt{x}$ .

13.  $\sqrt[3]{x - y}$ .

14.  $16\sqrt{2}$ .

15. 32.

## INVOLUTION OF RADICALS.

<b>Page 164.</b>	4. $18x$ .	7. $8ax^3\sqrt{a}$ .
1, 2. Given.	5. $8a$ .	8. $9b^2$ .
3. $a^{\frac{1}{2}}$ .	6. $\frac{x^4}{4}\sqrt{2x}$ .	9. $a^2 + 2a\sqrt{y} + y$ .

## EVOLUTION OF RADICALS.

<b>Page 165.</b>	<b>Page 166.</b>	8. $a - 5$ .
1. Given.	1-3. Given.	9. $3\sqrt{a} - \sqrt{8}$ .
2. $3\sqrt[3]{a}$ .	4. $a^{\frac{1}{2}}$ .	10. $4\sqrt{2a} + 5\sqrt{b}$ .
3. $2\sqrt[3]{3x}$ .	5. $a^{\frac{1}{2}}c^{\frac{1}{2}}$ .	<b>Page 168.</b>
4. $\sqrt[3]{9xy}$ .	6. $(a + b)^{\frac{1}{2}}$ .	1-3. Given.
5. $\sqrt[3]{8b^3}$ .	7. $\sqrt{ac}$ .	4. $\frac{c\sqrt[3]{x^3}}{x}$ .
6. $\sqrt[3]{a^2bc}$ .	8. $\sqrt[3]{x + y}$ .	5. $\frac{a\sqrt[3]{c^3}}{c}$ .
7. $\sqrt[3]{\frac{4}{9}}$ .	9. $\sqrt[3]{a + b}$ .	6. $\frac{x + 2\sqrt{xy} + y}{x - y}$ .
8. $ac^{\frac{1}{2}}$ .	10. $\sqrt{a + b + c}$ .	7. $\frac{x(\sqrt{a} + \sqrt{c})}{a - c}$ .
9. $2a^{\frac{1}{2}}$ .	<b>Page 167.</b>	8. $\frac{\sqrt{3} - 1}{2}$ .
10. $a^{\frac{1}{2}}b^{\frac{1}{2}}$ .	1, 2. Given.	9. $\frac{\sqrt{3} + 1}{2}$ .
11. $\sqrt{2a}$ .	3. $x - 4\sqrt{9}$ .	
12. $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$ .	4. 3.	
	5. $\sqrt{7} - \sqrt{a}$ .	
	6. 31.	
	7. $\sqrt{3a} + \sqrt{3b}$ .	

## RADICAL EQUATIONS.

<b>Pages 169, 170.</b>	9. $3^{\frac{1}{2}}$ .	15. $a\sqrt{\frac{1}{3}}$ .
1-3. Given.	10. 21.	16. Given.
4. $(d - a - c)^n$ .	11. 252.	17. 4.
5. 25.	12. $\frac{b^3 - 2ab^2 + a^2 - b}{2a}$ .	18. $\frac{2}{25}$ .
6. $4\frac{1}{2}$ .	13. Given.	19. $\frac{1}{1 - a}$ .
7. 256.	14. $\frac{1}{1 - a}$ .	
8. 1109.		

## PURE QUADRATICS.

*Page 173.*

1. Given.
2.  $x = \pm 5$ .
3.  $x = \pm 3$ .
4.  $x = \pm 4$ .
5.  $x = \pm 5$ .
6.  $x = \pm 4$ .
7.  $x = \pm 6$ .
8.  $x = \pm 4$ .
9.  $x = \pm \sqrt{6}$ .
10.  $x = \pm 7$ .
11.  $x = \pm 2$ .
12.  $x = \pm a$ .
13.  $x = \pm 1$ .

14.  $x = \pm 2$ .

15.  $x = \pm 3$ .

16.  $x = \pm 1$ .

17.  $x = \pm \frac{1}{a-1}$ .

18. Given.

19.  $x = \pm 3$ .

20.  $x = \pm 2a$ .

21.  $x = \pm \sqrt{c^3 + d^2}$ .

22.  $x = \pm \frac{1}{2}$ .

23.  $x = \pm \sqrt{a^3 + b^2}$ .

24.  $x = \pm 26$ .

*Page 174.*

1.  $x = \pm 36$ .

2.  $x = \pm 8$ .

3. 40 rods.

4. 30 rods.

5. 12, one;  
30, other.

6. \$6.

7. 80.

8. 15, less.  
60, greater.

9. 27 yds.;  
\$1.50, price

10.  $x = \pm 16$ .

11. 77 ft.

12.  $x = \pm 16$ .

## AFFECTED QUADRATICS.

*Pages 178, 179.*

- 1-5. Given.
6.  $x = 6$  or 2.
7.  $x = 9$  or -1.
8.  $x = 3a$   
 $\pm \sqrt{d + 9a^2}$ .
9. Given.
10. 15 or -4.
11. 20 or -7.
12. Given.
1. Given.
2. 10 or -7.
3. 2 or -5.
4. 3 or  $1\frac{2}{3}$ .
5. 4 or  $-1\frac{2}{3}$ .
6.  $x = 2$ .

7.  $-\frac{a}{2b}$

$\pm \sqrt{ab + d + \frac{a^2}{4b^2}}$

8.  $-2a$

$\pm \sqrt{b + 4a^2}$

9. 7 or -5.

10.  $3 \pm 2\sqrt{-1}$ .

11. 6 or -3.

12. 2 or -3.

13.  $\frac{b}{2c}$

$\pm \sqrt{bd - ch + \frac{b^2}{4c^2}}$

*Page 181.*

- 1, 2. Given.
3. 3 or  $-4\frac{1}{2}$ .
4. 5 or -6.
5.  $\frac{1}{2}$  or -2.
6. 2 or  $-\frac{1}{2}$ .
7. 4 or  $-4\frac{2}{3}$ .
8. 4 or -1.
9. 3 or  $-4\frac{2}{3}$ .
10. 9 or 6.
11. 4 or  $-3\frac{2}{3}$ .

*Page 182.*

1. 3 or 1.
2. 4 or 1.
3. 3 or  $\frac{1}{3}$ .

4. 2 or  $-12$ .
5.  $1\frac{1}{2}$  or  $\frac{3}{2}$ .
6. 11 or 3.
7.  $1\frac{2}{3}$  or  $-1\frac{1}{3}$ .
8.  $-\frac{1}{2}$  or  $-1\frac{2}{3}$ .
9. 4 or  $2\frac{1}{2}$ .
10.  $1 \pm \sqrt{-a^2 + 1}$ .
11.  $-m$   
 $\pm \sqrt{b^2 + m^2}$ .
12. 1 or  $-1\frac{1}{2}$ .
13. 1 or  $-28$ .
14. 10 or  $-\frac{7}{10}$ .
15.  $-\frac{1}{6}$  or  $-\frac{1}{2}$ .
16. 4 or  $-1$ .
17. 4 or  $-1\frac{3}{4}$ .
18. 5 or  $-4\frac{7}{8}$ .
19.  $1\frac{1}{2}$  or  $-\frac{1}{2}$ .
20. 4 or  $-1$ .
21.  $1\frac{1}{2}$  or  $-\frac{1}{2}$ .
22.  $4\frac{1}{2}$  or  $\frac{1}{2}$ .
23. 3 or  $-1\frac{1}{2}$ .
24. 4 or  $-1\frac{1}{4}$ .
25.  $\frac{3}{8}$  or  $\frac{1}{8}$ .
26.  $\frac{1}{2}$  or  $-1$ .
27.  $n \pm m$ .
28.  $3b$  or  $3a - 3b$ .

*Page 183.*

- 1-3. Given.
4.  $\pm 2$  or  $\pm \sqrt{2}$ .
5.  $\pm \sqrt{3}$  or  
 $\pm \sqrt{-1}$ .
6.  $\sqrt[3]{7} = 1.91 +$ .
7.  $\frac{1}{4}$  or  $-\frac{3}{4}$ .

8.  $\frac{1}{8}$  or  $-8$ .
9.  $4\frac{1}{4}$  or  $\frac{1}{4}$ .
10. 4 or  $-21\frac{1}{2}$ .

*Pages 184, 185.*

1. 8 or 4, one;  
4 or 8, other.
2. \$60 or \$40.
3. 6 or 4, one;  
4 or 6, other.
4. 16s., \$5 each.
5. 5 or  $-6\frac{1}{2}$ .
6. 16 scholars.
7. \$30 or \$20;  
\$20 or \$30.
8. 60 or 40, one;  
40 or 60, other.
9. 36 rds. length;  
28 " breadth
10. 20 in file;  
80 in rank.
11. 10 lambs.
12. 2 and 2.
13. 4 and 1.
14. 121 yds. long;  
120 " wide.
15. 6 m., A's rate;  
5 m., B's rate.
16. 120, A;  
80, B.
17. 42 and 6.
18. 4 lemons, A;  
6 " B.
19. 14 ft., length;  
10 " breadth.

20. 12 rows;  
15 trees in each
21. 52.
22. 20 persons.

*Page 186.*

23. 8 or  $-10$ , less;  
15 or  $-12$ , gr.
24. 16 and 20.
25. 50 and 25.
26. 121 and 25.
27. 12 ft., fore-w.;  
15 ft., hind-w.
28. 2 or  $-18$ , one;  
18 or  $-2$ , oth.
29.  $\frac{5}{8}$  and  $\frac{1}{4}$ .
30. 3, less;  
18, greater.
31. 16 or 36 yrs.
32. 28 rods, length;  
20 " breadth.
33. 15 yrs., A's;  
8 yrs., B's.
34. 20 lbs. pepper.

*Pages 188, 189.*

1. Given.
2.  $x = 4$  or 3;  
 $y = 3$  or 4.
3.  $x = 7$  or 5;  
 $y = 5$  or 7.
4.  $x = 8$ ,  $y = 6$ .
5.  $x = 10$  or  $-12$ ;  
 $y = 12$  or  $-10$ .
6.  $x = 10$ ;  
 $y = 12\frac{1}{2}$  or 7.

- |  |  |  |
|--|--|--|
| 7. $x = 9$ ;<br>$y = 4$ or $2\frac{2}{3}$ .  | 16. $x = 21$ or $-7$ ;<br>$y = 7$ or $-21$ .           | 4. 40 rows;<br>25 trees in each            |
| 9. $x = 5$ or $4$ ;<br>$y = 4$ or $5$ .      | 17. $x = 625$ ;<br>$y = 16$ .                          | 5. 40 yds, length;<br>24 " breadth.        |
| 10. $x = 5$ or $-3$ ;<br>$y = 3$ or $-5$ .   | 18. $x = 2$ or $1$ ;<br>$y = 1$ or $2$ .               | 6. 9 and 3.                                |
| 11. $x = 3$ ; $y = 2$ .                      |  | 7. 11 and 7.                               |
| 12. $x = \pm 5$ ;<br>$y = \pm 3$ .           | <b>Page 191.</b>                                       | 8. 31 rds, length;<br>19 " width.          |
|  | 1. 8 or $-4$ , gr.;<br>4 or $-8$ , less.               | 9. $\pm 7$ and $\pm 4$ .                   |
| <b>Page 190.</b>                             | 2. 30 yrs., wife;<br>31 " man.                         | 10. 25 m. and 23 m.                        |
| 15. $x = 15$ or $12$ ;<br>$y = 12$ or $15$ . | 3. $x = 9\sqrt{2}$ , gr.;<br>$y = \pm\sqrt{2}$ , less. | 11. 12 and 4.                              |
|  |  | 12. 3 or $-2$ , one;<br>2 or $-3$ , other. |

## RATIO.

- |                    |                      |                                     |
|--------------------|----------------------|-------------------------------------|
| <b>Page 195.</b>   | 10. $\frac{1}{4}$ .  | 19. Equality.                       |
| 3. $\frac{1}{4}$ . | 11. $\frac{a}{2}$ .  | 20. Equality.                       |
| 4. $\frac{1}{8}$ . | 12. $x - y$ .        | 21. Gr. inequality.                 |
| 5. $\frac{1}{8}$ . | 14. $\frac{4}{3}$ .  | 22. Less inequality.                |
| 6. $2a$ .          | 15. $\frac{2}{3x}$ . | 23. $\frac{3}{8} > \frac{1}{5}$ .   |
| 7. $3c$ .          | 17. $\frac{1}{4}$ .  | 24. $\frac{8}{15} < \frac{1}{11}$ . |
| 8. 10.             | 18. $\frac{1}{17}$ . | 25. 7.                              |
| 9. $\frac{1}{8}$ . |                      | 26. 98.                             |

## PROPORTION.

- |                  |                            |   |
|------------------|----------------------------|---|
| <b>Page 204.</b> | 5. 32 and 24.              | 10. 430 r., length;<br>320 r., breadth. |
| 2. 4.            | 6. 10 and 8.               | 11. 20 r.; 30 r.                        |
| 3. 6400.         | 7. 16 and 12.              | 12. 9 and 15.                           |
| 4. 12.           | 8. 6 and 4.                | 13. 20 and 16.                          |
|                  | 9. 48 and $9\frac{3}{8}$ . |   |

## ARITHMETICAL PROGRESSION

- |                  |                     |                       |
|------------------|---------------------|-----------------------|
| <b>Page 207.</b> | 4. $-5$ .           | 9. 15.                |
| 2. 9.            | 5. $1\frac{3}{4}$ . | 10. $44\frac{1}{2}$ . |
| 3. 68.           | 6. .91.             | 11. $49x$ .           |
|                  | 8. 43.              | 12. $3an - a$ .       |

**Page 208.**

2.  $762\frac{1}{2}$ .
3. 216.
4. 1400.
5.  $25\frac{1}{2}$ .
6. 610.
7. 175.
8. 810.

**Page 209.**

1. 58.
2. 278.
3. 11.
4. — 43.
5.  $2\frac{1}{2}$ .
6. —  $2\frac{1}{2}$ .
7. 1024.
8. 192.

**Page 210.**

1. 175.
2. 1130.
3. 6.
4. 6.
5. 259.

6. 13.
7. — 11.
8. 0.
9. 255.
10. 62.
11. 61.

**Page 212.**

1. 1, 7, 13, 19, 25, 31.
2.  $3, 7\frac{1}{2}, 12, 16\frac{1}{2}, 21, 25\frac{1}{2}, 30, 34\frac{1}{2}, 39, 43\frac{1}{2}, 48$ .
1. 47.
2. — 6.
3. 102.
4.  $2, 11\frac{2}{3}, 21\frac{1}{3}, 31, 40\frac{2}{3}, 50\frac{1}{3}, 60$ .
5.  $1683\frac{1}{3}$ .
6.  $98\frac{3}{4}$ .
7. 5776.
8. 10100.

**Page 213.**

9. 5.

10.  $6, 13\frac{1}{3}, 20\frac{2}{3}, 28, 35\frac{1}{3}, 42\frac{2}{3}, 50, 57\frac{1}{3}, 64\frac{2}{3}, 72$ .
11.  $12, 21.6, 31.2, 40.8, 50.4, 60, 69.6, 79.2, 88.8, 98.4, 108$ .

12. 975.
14. 3, 5, and 7.
15. 10100 yards, or  $5\frac{1}{2}$  mi., nearly.

**Page 214.**

16. 156.
17. \$62.50.
18. \$667.95.
19. 300.
20. \$1.20, int.; \$2.20, amt.
21. 20, 40, and 60.
22.  $16.61 +$  days.
23. 30, 40, 50, 60.
24. 3 days.
25. 140.
26. \$178, last pay't; \$5370, debt.

GEOMETRICAL PROGRESSION.

**Page 216.**

1. 160.
2. 4374.
3.  $4\frac{1}{2}$ .
4. 320.
5. 112.
6. — 31250.

**Pages 217-221.**

2. 11718.
3. 9999.
4. 27305.
5. 3885.
6. 8525.
1. 30000.
2. 15625.

3. 2.
4. 3.
5. 6.
6. 5.
1. 242.
2. 2.
3. 500.
4. 5.

5. 215.	6. 43046721.	<i>Page 223.</i>
6. 567.	8. \$4095.	14. \$120, \$60, \$30
2. $\frac{1}{2}$ , 2, 8, 32, 128.	9. \$196.83, l. c.;	15. 3, 15, 75, 375.
1. 4371.	\$295.24, wh. c.	1875.
2. $\frac{1649}{2187}$ .	10. \$10.23.	16. \$108, \$144,
3. 7174453.	11. 2, 6, 18.	\$192, \$256.
4. $\frac{2173954}{1144}$ .	12. \$4294967.295.	17. $\frac{11}{10}$ , or 1.1.
5. 9565938.	13. 10, 30, 90, 270.	18. 8, 4, 2, 1.

## INFINITE SERIES.

<i>Page 231.</i>	5. 2.	9. $\frac{2}{3}$ .
1. $1\frac{1}{2}$ .	6. 9.	10. $\frac{1}{a-1}$ .
2. $\frac{2}{3}$ .	7. 10.	11. 50 rods.
3. 1.	8. $\frac{1}{3}$ .	
4. $1\frac{1}{2}$ .		

## LOGARITHMS.

<i>Page 237.</i>	11. .1814.	20. 2.504.
2. 1548.86.	12. — 4.619.	21. 2.124.
3. 1.973.	<i>Page 239.</i>	<i>Page 240.</i>
4. III	14. .0003321.	23. .342+.
6. 78.	15. 33.335.	24. .546+.
7. .0375.	16. 191.77.	25. .324+.
8. 14.38.	18. 5.23.	26. Given.
9. 2.723.	19. 1.0836.	
10. 2906.3.		

## BUSINESS FORMULAS.

<i>Pages 245-260.</i>	17. \$2010.14.	29. \$8.83+.
2. \$349.60.	19. \$5414.28.	31. $5\frac{255}{100}$ per cent.
4. $16\frac{2}{3}$ per cent.	21. \$2769.23, pr. w.;	32. $12\frac{1}{2}$ per cent.
6. \$12600.	\$830.77, disc.	33. $9\frac{8}{13}$ per cent.
8. \$840.	22. \$6000, pr. w.;	34. 0. 8 per cents.
10. \$600.	\$1800, disc.	35. \$24630.54, in.;
12. $16\frac{2}{3}$ years.	24. \$1718.75.	\$369.46, com.
13. 10 years.	26. \$2125.	38. \$1332.
15. $2\frac{1}{2}$ per cent.	28. \$2.33 $\frac{1}{2}$ .	39. \$6290.15.

40. \$905.80.  
 41. \$3278.69.  
 42. \$2278.48.  
 44. \$6130.67.  
 45. \$2767.60.  
 47. \$2336.25.  
 49. \$14166.67.  
 51. \$14775.  
 53. \$249.77.

**Pages 265-268.**

2.  $+\sqrt{xy}$ .  
 3.  $-6$ .  
 4.  $6\sqrt{-1}$ .  
 5.  $\sqrt{-xy}$ .  
 7. 1.  
 8.  $\sqrt{\frac{-x}{y}}$ .

9.  $\sqrt{\frac{x}{-y}}$ .  
 10.  $5\sqrt{2}$ .  
 11.  $\frac{c}{d}$ .  
 2. Impossible.  
 3. Impossible.

## TEST EXAMPLES.

**Page 274.**

1. 75a.  
 2.  $57x + 7$ .  
 3.  $3ax + 3ab + 2cd$   
 4.  $7bc + 3cd$   
      $-10xy + 5mn$ .  
 5. —.  
 6. —.  
 7. —.  
 8. 8.  
 9. 31.  
 10.  $c(3b^2 - 6b^2c - cd)$ .  
 11.  $3x^2(y - 3z - 6yz)$   
 12.  $(a^n + b^n)(a^n - b^n)$   
 13.  $2 \times 2(2a - 1)$ .  
 14.  $(a^2 + 1)(a + 1)$   
      $\times (a - 1)$ .  
 15. 20, 11.  
 16. —.  
 17. 24 shots.

**Page 275.**

18.  $\pm\sqrt{ab}$ ;  
      $\pm\sqrt{\frac{a}{b}}$ .  
 19.  $\frac{a + 1}{b}$ .  
 20.  $\frac{1}{a - b}$ .  
 21.  $(3xy + 2z)$   
      $\times (3xy + 2z)$ .  
 22.  $(3b - c)(3b - c)$ .  
 23. —.  
 24. 640 rods.  
 25.  $29\frac{1}{4}$  miles.  
 26.  $\frac{b - 1}{b + 1}$ .  
 27. —.  
 28. —.  
 29. 1.  
 30.  $\frac{a + b}{a - b}$ .  
 31.  $\frac{3a^3 - a^4}{1 - a^4}$ .  
 32. 120.  
 33. \$50.

**Page 276.**

34. 6 hours.  
 35.  $x = 5$ ;  $y = 3$ .  
 36.  $x = 5$ ;  $y = 2$ .  
 37. —.  
 38.  $85\frac{1}{3}$  miles.  
 39.  $\frac{1}{8}$ .  
 40.  $8\frac{3}{4}$  g.;  $5\frac{1}{4}$  l.  
 41.  $x = 4$ ;  $y = 6$ ;  
      $z = 8$ .  
 42.  $x = 2$ ;  $y = 4$ ;  
      $z = 8$ .  
 43.  $x = \$40$  A,  
      $y = \$60$  B,  
      $z = \$80$  C.  
 44. 3 meters f. w.  
     6 " h. w.  
 45. 8 ft. one; 10 ft.  
 46. —.  
 47. 49 and 77.  
 48. 48 meters.  
 49. \$2.46 a meter.  
 50. —.  
 51. \$100 horse,  
     \$200 carriage.

52. —.

53. 65 hectares y.  
100 " elder.

54. 26.

55. \$5 l.; \$6 s.

56. 577 v.; 848 v.

57. 90 yrs. A; 45 y.  
B; 15 y. C.58.  $9\sqrt{3}$ .59.  $y\sqrt{1+a}$ .. *Page 278.*60.  $(x^4)^{\frac{1}{2}}$ ,  $(y^8)^{\frac{1}{2}}$ .61.  $\sqrt[3]{27(a-b)^8}$ .

62. 90 cts.

63. \$10; \$18.

64. 15 men.

65. 28 persons.

66. \$4 b. \$6 w.

67. 9.

68. a.

69.  $\frac{1}{1-a}$ .70.  $a + b$ .

71. 240 liters.

72. —.

73. 12 and 8.

*Page 279.*

74. \$180.

75.  $\frac{1}{11}$ .

76. \$90.

77. \$21.

78. 24 s. \$5 pr.

79. 81.

80. 45 m.; 105 m.

81. 80 A; 70 B.

82.  $\sqrt{x} - \sqrt{7}$ .83.  $\sqrt{3x} + \sqrt{3y}$ .84.  $d^2 + 6d - 2b^2d$   
 $+ 9 - 6b^2 + b^4$ .85. \$12000 Mayor;  
\$1200 Clerk.86. 216 g.; 30 $\frac{1}{2}$  l.*Page 280.*

87. 8 and 6.

88. 8 cts.

89. 10 days.

90. \$1407 B;  
\$469 A.91. \$50 cow;  
\$200 h.

92. 6 miles.

93. 100 ft.  $\times$  60 ft.94.  $7\frac{2}{3}$ ,  $12\frac{1}{3}$ , 17,  
21 $\frac{2}{3}$ , 26 $\frac{1}{3}$ .95. 637 $\frac{1}{2}$ .

96. 50 pair.

97. 18 and 14.

98. 32 h.; 75 l.

99. 10 y. B;  
30 y. M.

100. \$105.

*Page 281.*101. 8:43 $\frac{7}{11}$  o'clock102. 250 pp. A;  
320 " B.

103. 20 days.

104. 14 $\frac{2}{3}$  yrs.

105. \$30.

106. \$9000 whole  
\$4800 A;  
\$4200 B.107. 184 v. and  
185 v.

108. 24 d.; 48 d.

109. 45.

110. 6, 18, 54, 162.

111. 16 hats.

*Page 282.*

112. 97656250.

113. 10 ft.; 30 ft.;  
50 ft.

114. 95; 42; 82.

115. 2, 4, 8.

116. 12 weeks.

117. \$180; \$120.

118. \$30.

119. 45 A.; 55 A.

120. 14348906.

121. 125 p. 175 h

122. 375 men.

